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ADVANCED ARITHMETIC AND ELEMENTARY ALGEBRA AND MENSURATION.

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ADVANCED ARITHMETIC
AND
ELEMENTARY ALGEBRA AND MENSURATION

UNIVERSITY OF TORONTO LIBRARY

A TEXT-BOOK FOR SECONDARY SCHOOLS AND
STUDENTS PREPARING FOR PUBLIC EXAMINATIONS

BY

P. GOYEN

CHIEF INSPECTOR OF SCHOOLS, OTAGO, NEW ZEALAND

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PREFACE.

In my *Advanced Arithmetic* I adopted the methods of algebra without using its notation; in the present book I use both its methods and its notation. To make the notation intelligible to students who have not read algebra, I open with a chapter on literal symbolic representation and such other parts of algebra as are necessary and useful in the generalisation of arithmetical practice and in the solution of equations of the first and second degree. Less than this would be insufficient, and more superfluous, for nine-tenths of the boys and girls whose education is to be carried beyond the primary but not to the University stage. My treatment of equations of the second degree may seem to suffer from the omission of exercises in the factorisation of compound algebraic expressions; but the method of solving such equations by completing the square is very simple and never fails, and may, I hope, be regarded as adequate in a book that is intended to embrace only those departments of algebra that can be made of real service to arithmetic.

In my treatment of decimals I have given great prominence to the significance of index notation. This I have done partly because decimal operations cannot be adequately understood without a sound knowledge of index notation, and partly that the student may the better understand the nature of logarithms when he comes upon them in the succeeding chapter.

The book is based on the recommendations of the Mathematical Association. It will, I venture to believe, be found useful to teachers, to those who are studying for the teaching profession, and to those who, a numerous class in the colonies, have to work, if they work at all, without the aid of a teacher, and therefore need all the help a text-book can give them.

My thanks are due and are heartily given to the friends who have assisted me. I am especially indebted to Mr. M'Elrea, B.E., Rector of the Balclutha District High School, and to Mr. C. R. Richardson, B.A., one of my colleagues, without whose able and untiring assistance, ungrudgingly given, I could not have undertaken the work.

I also tender my thanks to Mr. S. J. Brading, B.A., who has been good enough to see the book through the press, a duty which, owing to distance from London, I could not myself discharge.

Finally, I wish to express my obligation to the Controller of His Majesty's Stationery Office for permission to reproduce the Mathematical Tables published by the Board of Education.

Should those using the book find in it errors or obscurities, I shall esteem it a favour if they will be so good as to send a note of them either to the publishers or to me.

P. GOYEN.

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TABLES OF WEIGHTS AND MEASURES.

AVOIRDUPOIS WEIGHT.

16 drams (dr.)	make	1 ounce (oz.)
16 ounces	,,	1 pound (lb.)
14 pounds	,,	1 stone (st.)
28 pounds or 2 stones	,,	1 quarter (qr.)
4 quarters or 8 stones 112 pounds	,,	1 hundredweight (cwt.)
20 hundredweight 2240 pounds	,,	1 ton

TROY WEIGHT.

24 grains (gr.)	make	1 pennyweight (dwt.)
20 pennyweights	,,	1 ounce
12 ounces	,,	1 pound (not used now)

This weight is used for gold, silver, and jewels.

7000 grains	make	1 pound avoirdupois
437½ grains	,,	1 ounce avoirdupois
480 grains	,,	1 ounce troy

MEASURES OF LENGTH.

	12 inches	make	1 foot (ft.)
	3 feet	,,	1 yard (yd.)
	22 yards	,,	1 chain (ch.)
	10 chains	,,	1 furlong (fur.)
	8 furlongs 80 chains 1760 yards	,,	1 mile (m.)
Also:	7·92 inches	make	1 link (lk.)
	100 links	,,	1 chain
	25 links 5½ yards	,,	1 pole or rod
	4 poles	,,	1 chain

The following are sometimes useful:

1 hand = 4 inches (used in measuring the heights of horses).
1 fathom = 6 feet (used in measuring great depths).
1 league = 3 miles.
1 degree of latitude = about 70 miles.

MEASURES OF SURFACE OR AREA.

	144 square inches	make	1 square foot (sq. ft.)
	9 square feet	,,	1 square yard (sq. yd.)
	484 square yards 10,000 square links	,,	1 square chain (sq. ch.)
	10 square chains 4,840 square yards 100,000 square links	,,	1 acre
	640 acres	,,	1 square mile
Also:	30¼ square yards	make	1 perch or sq. pole
	40 perches	,,	1 rood
	4 roods	,,	1 acre

MEASURES OF CUBIC CONTENT.

1728 cubic inches make 1 cubic foot (cub. ft.)
27 cubic feet make 1 cubic yard (cub. yd.)

METRIC MEASURES.

The measures most commonly used are printed in thick type.

MEASURES OF WEIGHT.

10 **milligrams** (mg.) = 1 **centigram** (cg.)
10 **centigrams** = 1 decigram (dg.)
10 decigrams = 1 **GRAM** (g.) = 15·43 grains (nearly)
10 **grams** = 1 decagram (Dg.)
10 decagrams = 1 hectogram (Hg.)
10 hectograms = 1 **kilogram** (Kg.) = 2·2 lbs. (nearly)
100 **kilograms** = 1 **quintal** (q.)
1000 **kilograms** = 1 **tonne** (t.)

MEASURES OF LENGTH.

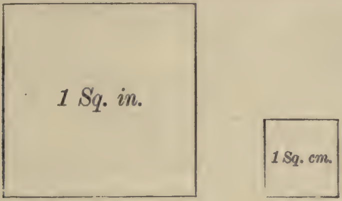
10 **millimetres** (mm.) = 1 **centimetre** (cm.)
10 **centimetres** = 1 decimetre (dm.)
10 decimetres = 1 **METRE** (m.) = 39·37 inches (nearly)
10 **metres** = 1 decametre (Dm.)
10 decametres = 1 hectometre (Hm.)
10 hectometres = 1 **kilometre** (Km.) = 5 furlongs (nearly)

MEASURES OF AREA.

100 sq. **millimetres** (sq. mm.) = 1 sq. **centimetre** (sq. cm.)
100 sq. **centimetres** = 1 sq. **decimetre** (sq. dm.)
100 sq. **decimetres** = 1 sq. **metre** (sq. m.) = 1 **centiare**
100 sq. metres = 1 sq. **decametre** (sq. Dm.) = 1 **are**
100 sq. decametres = 1 sq. hectometre (sq. Hm.) = 1 **hectare**
100 sq. hectometres = 1 sq. kilometre (sq. Km.)

Also for large areas :

100 centiares = 1 **ARE** (a.)
100 ares = 1 **hectare** (Ha.)
100 hectares = 1 sq. **kilometre** (sq. Km.)



MEASURES OF VOLUME.

1000 cubic millimetres (c.mm.) = 1 **cubic centimetre** (c.cm.)
1000 cubic centimetres = 1 **cubic decimetre** (c.dm.)
1000 cubic decimetres = 1 **cubic metre** (c.m.) = 1 **stere**

MEASURES OF CAPACITY.

10 centilitres (cl.) = 1 **decilitre** (dl.)
10 decilitres = 1 **LITRE** (l.) = $1\frac{3}{4}$ pints (nearly)
10 litres = 1 **decalitre** (Dl.)
10 decalitres = 1 **hectolitre** (Hl.) = 22 gallons (nearly)

The litre is the unit of capacity. It is equal to the volume of a cubic decimetre, that is, of a cube each edge of which is 10 centimetres in length. The volume of a litre is therefore 1000 cubic centimetres, and the weight of a litre of pure water at a temperature of 4° C. is 1000 grams ; hence the weight of 1 cubic centimetre of pure water at this temperature is 1 gram.

BRITISH MEASURES OF CAPACITY.

4 gills (nearly obsolete)	make	1 pint
2 pints	,,	1 quart
4 quarts	,,	1 gallon
2 gallons	,,	1 peck (pk.)
4 pecks or 8 gallons	,,	1 bushel (bus. or bush.)
8 bushels	,,	1 quarter (qr.)

It may be mentioned that in the colonies dry goods are generally sold by weight, not by measure.

1 bushel of oats is reckoned to weigh 40 lbs., and 1 bushel of wheat 60 lbs.

The volume of a gallon is 277·274 (approximately $277\frac{1}{4}$) cubic inches, and a gallon of pure water at a temperature of 62° F., barometer at 30 inches, weighs 10 lbs.

One cubic foot of pure water has a volume of about $6\frac{1}{4}$ gallons, and weighs about 62·3 lbs. or nearly 1000 ounces.

MEASURES OF TIME.

60 seconds (sec.)	make	1 minute (min.)
60 minutes	,,	1 hour (hr.)
24 hours	,,	1 day (da.)
7 days	,,	1 week (wk.)
52 weeks	,,	1 year (yr.)
365 days	,,	1 common year
366 days	,,	1 leap year
100 years	,,	1 century

A *solar year* is the time taken by the earth to make a complete revolution round the sun, and contains 365¼ days[*]; but, to avoid counting the quarters every year, the year is considered to be 365 days; and because in four years the quarters will amount to four, or one whole day, every fourth year is reckoned to contain 366 days, and is called *leap year*. The extra day of leap year is added to February. With very rare exceptions[†] leap years are those the number of which is divisible by 4. The year is divided into 12 calendar months, which are of different lengths. Seven of them contain 31 days, four 30 days, and February in the common year has 28. The number of days in each month is easily remembered from the following lines:

"Thirty days have *September*,
April, *June*, and *November*.
February has twenty-eight alone,
And all the rest have thirty-one;
But leap year coming once in four,
February then has one day more."

ANGULAR MEASURE.

60 seconds (″)	make	1 minute (′)
60 minutes	,,	1 degree (°)
90 degrees	,,	1 right-angle (∟)
360 degrees	,,	1 circle or revolution (⊙)
1 degree of longitude	,,	4 minutes' difference in time

MISCELLANEOUS.

24 sheets of paper	make	1 quire
20 quires	,,	1 ream
10 reams	,,	1 bale
12 units	,,	1 dozen
12 dozen or 144 units	,,	1 gross
20 units	,,	1 score

---

[*] 365 days 5 hrs. 48 mins. 48 secs.

[†] The exceptions are the centuries, the numbers of which are not divisible by 4. Thus 1800, 1900, 2100 are not leap years, for 18, 19, 21 are not divisible by 4.

CHAPTER I.

ON THE USE OF LITERAL SYMBOLS.

ARITHMETIC and ALGEBRA are alike in this: they both treat of quantities denoted by symbols. In arithmetic the symbols are figures; in algebra they are figures and letters.

In arithmetic the sum of 2 and 3 is thus denoted: $2+3$; in algebra the sum of a and b is denoted by $a+b$; but, while in arithmetic we can carry the expression to the definite result 5, we cannot, without assigning numerical values to a and b, carry the sum of a and b beyond the form $a+b$. If, however, we assign to a the value 9 and to b the value 10, we can write

$$a+b=9+10=19.$$

In arithmetic therefore the symbols are definite in value; but in algebra the literal symbols, as the letters are called, may have any values we choose to assign to them.

The signs $+$, $-$, $\times$, $\div$ have the same meaning in algebra as in arithmetic. It should be noted, however, that the sign $\times$ is seldom placed between letters, which are considered as multiplied when they are placed side by side, as:

$a\times b\times c=abc$; and a similar remark applies to the multiplication of letters by figures, as:

$7\times a=7a$, which means $a+a+a+a+a+a+a$, that is, a taken seven times.

Multiplication is sometimes denoted by the sign . placed between the factors, as $7\,.\,5=7\times 5$; $a\,.\,b=a\times b$; and division is, both in algebra and in arithmetic, sometimes denoted thus: $3/5$, a/b, x/y, meaning $3\div 5$, $a\div b$, $x\div y$.

Instead of $(4+3)\times 5$, we may write $5(4+3)$ or $(4+3)5$, forms which mean 5 times the sum of 4 and 3 or 5 times $4+5$ times 3.

In the same way, for $(4-3)\times 5$ we may write $5(4-3)$ or $(4-3)5$, each of these forms meaning 5 times the difference between 4 and 3 or 5 times 4 minus 5 times 3.

And always for all values of m, a, and b, the forms:

$$m(a+b)=ma+mb,$$
$$m(a-b)=ma-mb,$$

express the facts.

Suppose we have to find the numerical value of

$$ad+bc-b-a-de$$

when $a=1$, $b=2$, $c=3$, $d=4$, and $e=0$

$$\begin{aligned} ad+bc-b-a-de &= 1\times 4+2\times 3-2-1-0 \text{ (for } 4\times 0=0) \\ &= 4+6-3 \\ &= 7. \end{aligned}$$

EXAMPLES. I.

When the letters have the values given above, find the values of:

1. $a+c+cd$.
2. $bc-ad+de$.
3. $abc-4a+dc$.
4. $abcd+de-3a$.
5. $bce+cd+7c$.
6. $\frac{bc}{a}+\frac{cd}{ab}-5e$.
7. $\frac{abc}{2a}-4a+\frac{cd}{2b}$.
8. $\frac{bcd}{2d}-9e+\frac{de}{b}$.
9. $\frac{4bd}{2b}+3abc-9a$.
10. $\frac{9cd}{3ac}-\frac{7ad}{ab}-de$.

Quantities before which the minus sign $-$ is placed are called *minus or negative quantities.* Read off the negative quantities in the foregoing exercise.

If we have to subtract $5+3$ from 14, the minus quantity is $(5+3)$, which, expressed with its proper sign, is written thus:

$$-(5+3),$$

and means that the sum of 5 and 3 is to be taken from 14. Obviously, it is immaterial whether we take 8 from 14, or first 5 from 14 and then 3 from the remainder 9; for in each case the result is the same, namely, 6. In the latter case the work assumes the following form:

$$\begin{aligned} 14-(5+3) &= 14-5-3 \\ &= 6. \end{aligned}$$

The point to be carefully noted here is that, when the brackets are removed from the negative bracketed expression, the signs of the terms within the brackets are changed.

If we have to subtract $5-3$ from 14, the negative quantity is $(5-3)$, which, expressed with its proper sign, is written thus:

$$-(5-3),$$

and means that the difference between 5 and 3 is to be taken from 14. Hence $14-(5-3)=14-2=12$.

The working of such examples generally assumes the following form:

$$\begin{aligned} 14-(5-3) &= 14-5+3 \\ &= 9+3 \\ &= 12, \end{aligned}$$

where it is seen that subtracting 5 from 14 and adding 3 to the remainder is the same in effect as subtracting from 14 the excess of 5 over 3.

Again the point to be carefully noted is that, when the brackets are removed from the negative bracketed expression, the signs of the terms within the brackets are changed.

From a take the sum of b and c.

The quantity to be subtracted is $(b+c)$. Hence the answer is $a-(b+c)$ or, with the brackets removed, $a-b-c$.

From p take $(q-r-s)$.

The answer is $p-(q-r-s)$ or $p-q+r+s$.

And always, *when an expression within brackets is preceded by the sign* –, *the brackets may be removed if the sign of every term within the brackets be changed.*

The case of the bracketed expression preceded by the sign +.

$$\begin{aligned} 7+(8+5) &= 7+8+5; \\ 7+(8-5) &= 7+8-5; \\ a+(b+c) &= a+b+c; \\ a+(b-c) &= a+b-c. \end{aligned}$$

Hence, *when a bracketed expression is preceded by the sign* +, *no change is made in the signs of the terms within the brackets when the brackets are removed.*

EXAMPLES. II.

(*a*) Find the values of the following when $a=7$, $b=6$, $c=5$, $d=4$:

1. $a+(b-c)$.
2. $a-(b-c)$.
3. $4a-(b-c+d)$.
4. $3a-\left(b-2c-\frac{d}{2}\right)$.
5. $2a+(b-c-d)$.
6. $7b-(a-2c-2d)$.
7. $6c+\left(2a-5c+\frac{d}{4}\right)$.
8. $5d-(a-3c-3b)$.

(*b*) Write in words the meaning of:

1. $x+(y+z)$.
2. $x+(y-z)$.
3. $x-(y+z)$.
4. $x-(y-z)$.

In the expressions $7ab$, $9ab$, $10ab$, the numbers 7, 9, and 10 are called *numerical coefficients* of ab, that is, they are *numerical cofactors* of the products. Each of the letters is a *literal coefficient.*

A collection of algebraic symbols is called an *algebraic expression.* If the expression contains parts connected by the signs + and −, it is called a *compound expression*; and the parts connected by the signs + and − are called *terms* of the expression. Thus, $ab+cd-ef-xy$ is a compound expression, and ab, cd, ef, and xy are its terms, ab and cd being *positive terms* and ef and xy *negative terms.*

NOTE. When no sign precedes a term the sign + is understood.

If the terms of an expression differ only in their numerical coefficients, they are said to be *like* terms; but, if they differ in their letters, they are said to be *unlike* terms. Thus, in the expression $4ab-3cd+5pq-3xy+5cd+6ab$, $4ab$ and $6ab$ are *like* terms, and so are $3cd$ and $5cd$; while $5pq$ and $3xy$ are *unlike* terms.

Like terms can, by addition or subtraction or both, be expressed as a *single* like term; for they denote so many or so much of the same kind of thing. For example:

$$4ab-3cd+5pq-3xy+5cd+6ab=10ab+2cd+5pq-3xy.$$

The unlike terms $5pq-3xy$ remain unchanged; for they are incapable of simpler expression.

The operation we have just performed is called collecting or summing the terms. The following are the rules to be observed in this operation:

(1) When all the like terms are positive, add the numerical coefficient, as:

The sum of $4ab$ and $6ab = 10ab$.

(2) When all the like terms are negative, add the numerical coefficients and prefix the minus sign to the sum, as:

The sum of $-2p$, $-p$, $-3p = -6p$.

(3) When the like terms are not all of the same sign, add the numerical coefficients of the positive terms and then the numerical coefficients of the negative terms; take the difference of these two results, and prefix to the remainder the sign of the greater of them, as:

The sum of $3a$, $-2a$, $7a$, $-14a = 10a - 16a = -6a$.

EXAMPLES. III.

Collect the terms of the following:

1. $3b - 5b + 4b - b$.
2. $5c - 9c + 8c - 5c$.
3. $4ab + 3ab - 10ab$.
4. $5x - x - 3x + 7 - 5$.
5. $(x-4)3 + (2x-2)2$.
6. $(7x-3)4 - (2x-3)4$.
7. $(8x-5)5 + (3-2x)4$.
8. $(3x-2-x)6 - (3+2x-2)3$.

Subtraction further considered.

Ex. 1. *From* $3a - 2b + c$ *subtract* $-2a + b - c$.

The quantity to be subtracted may for clearness be bracketed thus: $(-2a + b - c)$;

$\therefore$ the difference of the quantities

$$= 3a - 2b + c - (-2a + b - c)$$
$$= 3a - 2b + c + 2a - b + c$$
$$= 5a - 3b + 2c \text{ (the terms collected).}$$

Ex. 2. *From* $3a - 2b + c$ *subtract* $2a + b - c$.

The quantity to be subtracted is $(2a + b - c)$;

$\therefore$ the difference of the quantities

$$= 3a - 2b + c - (2a + b - c)$$
$$= 3a - 2b + c - 2a - b + c$$
$$= a - 3b + 2c \text{ (the terms collected).}$$

In both examples we have simply changed the signs of the terms to be subtracted, and then collected the terms as in addition.

The rule for subtraction is: *Change the signs of all the terms to be subtracted, and then collect the terms as in addition.*

EXAMPLES. IV.

1. From $5a-2b-c$ subtract $3a-3b+3c$.

2. From $3x-4y+2z$ subtract $-x-3y-z$.

3. From $7p+q+4r$ subtract $7p-2q-5r$.

4. From $2x^2-3xy-4y^2$ subtract $-2x^2-2xy+3y^2$.

Multiplication.

To find the product of two algebraic expressions, we multiply each term of the multiplicand by each term of the multiplier.

Ex. 1. *Find the product of* $2ab$ *and* $3ad$.

$$2ab \times 3ad = 6a^2bd.$$

Ex. 2. *Find the product of* $(2a-3c)$ *and* $(3a+2d)$.

$$\begin{aligned}(2a-3c)(3a+2d) &= 3a(2a-3c)+2d(2a-3c)\\ &= 6a^2-9ac+4ad-6cd.\end{aligned}$$

Ex. 3. *Find the product of* $(2a-3c)$ *and* $(3a-2d)$.

$$\begin{aligned}(2a-3c)(3a-2d) &= 3a(2a-3c)-2d(2a-3c)\\ &= 6a^2-9ac-4ad+6cd.\end{aligned}$$

EXAMPLES. V.

Multiply

1. $a-b^2$ by ax.

2. $ab+2cd$ by xy.

3. $2a-b$ by $a-b$.

4. $2xy+3y$ by $7x-3y$.

5. $6x+4y$ by $2x-3y$.

6. $1-2mn$ by $2m+n$.

7. a^2-bc by $ac-b^2$.

8. $1+2x+3y$ by $x-y$.

9. $4x^2-6x+9$ by $2x+3$.

10. $4+2x+x^2$ by $4-2x+x^2$.

CHAPTER II.

EQUATIONS.

When two equal quantities are connected by the sign =, they are said to form an equation, as

$$7x=14;$$

$$\therefore \quad x=\tfrac{14}{7}=2.$$

$$14x=28;$$

$$\therefore \quad x=\tfrac{28}{14}=2.$$

The truth of these equations depends on the condition that the value of x is 2. The equations are therefore called *conditional equations.* It is in this sense that the word equation is generally used. The following,

$$x+y=y+x,$$

takes the form of an equation; but, since it is true for all values of x and y, it is called an *identity.*

In the working of equations, the following *axioms* are of great importance:

(1) *If equals be added to equals, the sums are equal.*

Thus, if $x=a$, $\begin{cases} \text{then} & x+4=a+4, \\ \text{and} & x+b=a+b. \end{cases}$

(2) *If equals be taken from equals, the remainders are equal.*

Thus, if $x=a$, $\begin{cases} \text{then} & x-4=a-4, \\ \text{and} & x-b=a-b. \end{cases}$

(3) *If equals be multiplied by equals, the products are equal.*

Thus, if $x=a$, $\begin{cases} \text{then} & 3x=3a, \\ \text{and} & bx=ab. \end{cases}$

(4) *If equals be divided by equals, the quotients are equal.*

$$\text{Thus, if } x=a, \begin{cases} \text{then} & \dfrac{x}{5}=\dfrac{a}{5}, \\ \text{and} & \dfrac{x}{b}=\dfrac{a}{b}. \end{cases}$$

Prove the above by substituting numbers for the letters.

As we have seen, an equation consists of two parts, one of which is placed to the right and the other to the left of the sign of equality. These are spoken of as the *sides* of the equation, the one on the left being called the *left side*, and the one on the right the *right side*.

The letter the value of which is to be found is called the *unknown quantity*, which is usually denoted by x. It may, of course, be denoted by any letter.

The process of finding the value of the unknown quantity is called *solving* the equation, and the value found is called the *root* or *solution* of the equation. Careful consideration of the working of the following examples will discover the nature of the process of finding the value of the unknown quantity that *satisfies* the equation.

Ex. 1. *Find the value of x that satisfies the equation $5x+9=24$.*

$$5x+9=24;$$
$$\therefore\ 5x=24-9; \qquad \text{[9 taken from each side. Axiom 2.]}$$
$$\therefore\ 5x=15;$$
$$\therefore\ x=\tfrac{15}{5}=3. \qquad \text{[both sides} \div 5. \text{ Axiom 4.]}$$

[PROOF. Put 3 for x; then $5\times3+9=24$.]

Ex. 2. *Solve the equation $4x-7=2x+3$.*

$$4x-7=2x+3;$$
$$\therefore\ 4x=2x+3+7; \qquad \text{[7 added to each side. Axiom 1.]}$$
$$\therefore\ 4x-2x=3+7; \qquad \text{[}2x\text{ taken from each side. Axiom 2.]}$$
$$\therefore\ 2x=10;$$
$$\therefore\ x=\tfrac{10}{2}=5. \qquad \text{[both sides} \div 2. \text{ Axiom 4.]}$$

[PROOF. Put 5 for x; then $4\times5-7=2\times5+3$.]

Ex. 3. *Find the value that satisfies x in $4(2x-5)=3x-(3-2x)-5$.*

$$4(2x-5)=3x-(3-2x)-5;$$

$$\therefore\ 8x-20=3x-3+2x-5;$$

$$\therefore\ 8x-5x-20=-3-5; \quad \text{[5}x\text{ taken from each side. Axiom 2.]}$$

$$\therefore\ 3x=20-3-5 \quad \text{[20 added to each side. Axiom 1.]}$$

$$=12;$$

$$\therefore\ x=\tfrac{12}{3}=4. \quad \text{[both sides} \div 3. \text{ Axiom 4.]}$$

[Proof. Put 4 for x; then $8\times4-20=3\times4-3+2\times4-5$.]

Ex. 4. *Solve*: $2x-3=5x-21$.

$$2x-3=5x-21;$$

$$\therefore\ 2x-5x=-21+3; \quad \text{[5}x\text{ taken from each side and 3 added to each side.]}$$

$$\therefore\ -3x=-18,$$

that is, $3x=18$; [the signs of both sides changed.]

$$\therefore\ x=\tfrac{18}{3} \quad \text{[both sides} \div 3.]$$

$$=6.$$

[Proof. Put 6 for x; then $2\times6-3=5\times6-21$.]

From the working of these examples we may enunciate the following rules for the solution of equations:

(1) *Transpose the terms containing the unknown quantity to the left hand side of the equation.*

(2) *Transpose the other terms to the right-hand side.*

(3) *Change the sign of every term transposed from one side to the other.*

(4) *When each side is reduced to one term, divide both sides by the numerical coefficient of the unknown quantity.*

Note. *If the sign of every term of an equation be changed, the equality of the sides still holds.* This is exemplified in the last example worked above.

Ex. 5. *Given that $ab=c+d$, to find the value of a and of b.*

$$ab=c+d;$$

$$\therefore\ a=\frac{c+d}{b},$$

$$\text{and } b=\frac{c+d}{a}.$$

Ex. 6. *Given that* $xy+c=a+b-c$, *to find the value of* x *and of* y.

$$xy+c=a+b-c;$$
$$\therefore\ xy=a+b-2c;$$
$$\therefore\ x=\frac{a+b-2c}{y},$$
$$\text{and }\ y=\frac{a+b-2c}{x}.$$

EXAMPLES. VI.

Solve:

1. $17x=153$.
2. $4x+17=12x-7$.
3. $5(x-2)=2(x+4)$.
4. $6(2x-3)=3(3x-2)+3$.
5. $8(\frac{1}{2}x+9)=12(x-\frac{1}{2})-2$.
6. $7x-(9-x)=14x-(45-2x)$.
7. $6(x+17)-9x-75=0$.
8. $2(x-5)+x+10=10(x-5)-2x$.
9. $8(\frac{1}{4}x+2)-(x+12)=4(\frac{1}{2}x-2)$.
10. $3(x+5)-2(x+13)=0$.

EXAMPLES. VII.

Find the value of x and of y in:

1. $xy=r+q$.
2. $xy=abc$.
3. $2xy=ab-c$.
4. $xy-a=2a+b$.
5. $xy-b=b-c+d$.
6. $xyz=pqr$.
7. $3xy-a=2a+b$.
8. $5xy+b=7ab-b$.

Compare the work under (A) with the corresponding work under (B) in:

	(A)	(B)
(1)	£4 = (4×20) shillings = (4×240) pence.	£x = $20x$ shillings = $240x$ pence.
(2)	400 pence = $\frac{400}{12}$ shillings = £$\frac{400}{240}$.	x pence = $\frac{x}{12}$ shillings = £$\frac{x}{240}$.
(3)	£2. 10*s*. 0*d*. = $(2\times 20+10)$ shillings.	£$x+y$ shillings = $(20x+y)$ shillings.
(4)	9 yds. = (9×3) ft. = (9×36) inches.	x yds. = $3x$ ft. = $36x$ inches.
(5)	150 inches = $\frac{150}{12}$ ft. = $\frac{150}{36}$ yds.	x inches = $\frac{x}{12}$ ft. = $\frac{x}{36}$ yds.
(6)	5 cwt. 2 qrs. = $(5\times 4+2)$ qrs.	x cwt. y qrs. = $(4x+y)$ qrs.

The following groups of questions are intended to show how literal symbols are used to express the facts of arithmetic *in a generalised way.* Before working the questions, note carefully that

$$a-b$$

may be read as:

(1) a minus b (that is, the quantity denoted by a minus the quantity denoted by b);

(2) the difference between a and b;

(3) what is left after b is taken from a;

(4) how much greater a is than b;

(5) how much must be added to b to make a;

(6) how much less b is than a;

(7) the defect of b from a;

(8) the excess of a over b.

EXAMPLES. VIII.

1. (*a*) How much is 5 greater than 3?
(*b*) How much is p greater than q?

2. (*a*) From the sum of 4 and 7 take their difference.
(*b*) From the sum of a and b take their difference, a being the greater.

3. (*a*) To the sum of 3 and 6 add their product.
(*b*) To the sum of c and d add their product.

4. (*a*) To the excess of 9 over 4 add the defect of 3 from 5.
(*b*) To the excess of x over y add the defect of a from b.

5. (*a*) Find the sum of 3 and the quotient of 10 divided by 5.
(*b*) Find the sum of b and the quotient of d divided by c.

6. (*a*) What is the double of 9? the double of $2+3$?
(*b*) What is the double of y? the double of $x+y$?

7. (*a*) By what must 4 be multiplied to make 16?
(*b*) By what must a be multiplied to make b?

8. (*a*) Multiply the sum of 4 and 5 by 6.
(*b*) Multiply the sum of x and y by z.

9. (*a*) How much must be added to 9 to make 12?
(*b*) How much must be added to d to make c?

10. (*a*) The product of two numbers is 16, and one of them is 8: find the other.

(*b*) The product of two quantities is p, and one of them is q: find the other.

11. (*a*) 24 is divided into two parts, one of which is 15: find the other part.

(*b*) x is divided into two parts, one of which is y: find the other part.

12. (*a*) The sum of 3 numbers is 18; one of them is 5, and another 3: find the third number.

(*b*) The sum of 3 numbers is a; one of them is b, and another is c: find the third.

13. (*a*) A boy is 12 years old: how old will he be 5 years hence?

(*b*) A boy is x years old: how old will he be y years hence?

14. (*a*) A boy is 13 years old: how old was he 5 years ago?

(*b*) A boy is p years old: how old was he q years ago?

15. (*a*) How old is a girl who was 11 years old 5 years ago?

(*b*) How old is a girl who was x years old y years ago?

16. (*a*) 11 is an odd number: what are the odd numbers next above and below it?

(*b*) p is an odd number: what are the odd numbers next above and below it?

17. (*a*) 12 is an even number: what are the even numbers next above and below it?

(*b*) a is an even number: what are the even numbers next above and below it?

18. (*a*) A father who is 5 times as old as his son is 30 years old: how old is the son?

(*b*) A father who is x years old is y times as old as his son: how old is the son?

19. (*a*) How long will it take a person to walk 15 miles if he walks 3 miles an hour?

(*b*) How long will it take a person to walk x miles if he walks y miles an hour?

20. (*a*) A man walks 4 miles an hour: how far does he walk in 5 hours?

(*b*) A man walks p miles on hour: how far does he walk in q hours?

21. (*a*) I buy 7 things at 2*s*. each: how much do I pay for them?

(*b*) I buy x things at y shillings each: how much do I pay for them?

22. (*a*) Divide the sum of 7 and 5 by their difference.

(*b*) Divide the sum of x and y by their difference, x being the larger.

23. (*a*) If a boy saves 3 shillings a week, how long will it take him to save 18 shillings?

(*b*) If a boy saves m shillings a week, how long will it take him to save n shillings?

24. (*a*) If I buy 10 eggs for 15 pence, what is the price of 1 egg?

(*b*) If I buy x eggs for y pence, what is the price of 1 egg?

25. (*a*) A boy begins to play with 12 marbles; he loses 4 and gains 5: how many marbles has he at the end of the play?

(*b*) A boy begins to play with x marbles; he loses y marbles and gains z marbles: how many has he at the end of the play?

26. How many inches in y yards? furlongs in x miles? shillings in x threepences? pence in £$x+y$ shillings $+z$ pence?

27. At 2*s.* each, how many things can be bought with £$a+b$ shillings?

28. Find in yards, the perimeter of a square the side of which is x chains.

29. How many lbs. of sugar at r pence per lb. can be bought with q half-crowns?

30. How many fowls at x shillings each can be bought with £y?

SOLUTION OF PROBLEMS.

Ex. 1. *The sum of two numbers is* 395, *and the larger is four times the smaller: find the numbers.*

Let x represent the smaller number; then $395-x$ represents the larger number, and 4 times the smaller number = the larger;

$$\therefore\ 4x=395-x;$$
$$\therefore\ 5x=395;$$
$$\therefore\ x=\frac{395}{5}=79;$$
$$\therefore\ \text{the larger number}=79\times4=316.$$

[Proof. Put 79 for x; then $4\times79=395-79$.]

Ex. 2. *After spending* £2 *more than half my money, I had* £6 *left: how much had I at first?*

Let x represent my money in pounds; then the money I spent is $\frac{1}{2}x+£2$;

$$\therefore\ \text{the remainder}=x-(\tfrac{1}{2}x+£2);$$
$$\text{but the remainder is } £6;$$
$$\therefore\ x-(\tfrac{1}{2}x+£2)=£6;$$
$$\therefore\ x-\tfrac{1}{2}x=£6+£2=£8;$$
$$\therefore\ \tfrac{1}{2}x=£8;$$
$$\therefore\ x=£16\ (\text{both sides}\times2).$$

[Proof. Put £16 for x; then $£16-£8-£2=£6$.]

Ex. 1. *Divide £18 into two parts such that twice the greater may exceed three times the less by £6.*

Let x represent the greater in pounds; then £18 − x represents the less;

$$\therefore\ 2x = 3(£18 - x) + £6;$$
$$\therefore\ 2x = £54 - 3x + £6;$$
$$\therefore\ 5x = £60;$$
$$\therefore\ x = \frac{£60}{5} = £12,$$

and the less = £18 − £12 = £6.

[Proof. Put £12 for x; then $£12 \times 2 = £18 \times 3 - £12 \times 3 + £6$.]

EXAMPLES. IX.

1. If 24 be added to a number the sum is 4 times the original number; find the original number.

2. Divide £150 into two parts such that the one is 4 times the other.

3. Divide a line 2 feet 8 inches long into two parts such that one is 3 times as long as the other.

4. The difference between A's and B's money is £15, and A has 3 times as much as B: find how much each has.

5. Find two numbers such that their sum is 25, and their difference 9.

6. A father is 4 times as old as his son; in 16 years he will be twice as old; find their ages.

7. A and B have between them £56, and 4 times A's money is equal to 3 times B's: find how much each has.

8. A party pays 5 shillings each for the hire of a conveyance; if there had been 4 more in the party, the cost to each of it would have been 3 shillings each: find how many were in the party.

9. I pay a bill with 8 coins, of which some are half-crowns and the rest florins: if the total value of the coins is 17*s.* 6*d.*, how many of each coin are there?

10. A bill of £4. 15*s.* was paid with half-sovereigns and half-crowns, the number of coins being 17: find how many of each coin were used.

11. I have a certain number of apples and half as many pears, and the apples and pears together number 39: how many of each have I?

12. I buy 16 lbs. of tea for 19*s.* 5*d.*, some of it at 1*s.* 2*d.* a lb. and the rest at 1*s.* 3*d.* per lb.: how many lbs. of each do I buy?

Equations involving two or more unknown quantities.

Consider the equation $x + y = 10$. ....(1)

$$x + y = 10;$$
$$\therefore\ x = 10 - y.$$

Hence, for every value we give to x, we get for y a corresponding value that will satisfy the equation.

If $x=1$, then $y=9$;
if $x=2$, then $y=8$;
if $x=\frac{1}{2}$, then $y=9\frac{1}{2}$;

and so on indefinitely.

But, if we have a second equation of the same kind involving x and y, as

$$2x-3y=5, \text{.................................}(2)$$

then we have $2x=5+3y$,

where $x=\frac{5+3y}{2}$.

If now we are required to find for x and y values that satisfy both equations, the value of x and of y must be identical in both.

In the first, $x=10-y$,

and in the second, $x=\frac{5+3y}{2}$;

$$\therefore \frac{5+3y}{2}=10-y;$$

$$\therefore 5+3y=20-2y;$$

$$\therefore 5y=15;$$

$$\therefore y=3.$$

Substituting this value for y in the first equation, we have

$$x=10-3=7;$$

and, substituting it for y in the second equation, we have

$$x=\frac{5+9}{2}=7.$$

[PROOF. $7+3=10$, and $14-9=5$.]

And, generally, if there be two independent equations involving the same two unknown quantities, they can be reduced to one equation involving only one of the unknown quantities. Such equations are called *simultaneous equations.*

The process by which we get rid of one of the unknown quantities is called *elimination.*

Ex. 1. *Given*

$$\left.\begin{array}{ll}(1) & 3x-5y=13\\(2) & 2x+7y=81\end{array}\right\}$$

to find the values of x and y that satisfy both equations.

To eliminate x, multiply (1) by 2 and (2) by 3.

$$\begin{array}{rl}\text{Then} & 6x-10y=26 \quad\ldots\ldots\ldots\ldots(1)\\ \text{and} & \underline{6x+21y=243} \quad\ldots\ldots\ldots\ldots(2)\\ & 31y=217 \text{ [subtracting (1) from (2)];}\\ & \therefore\ y=7.\end{array}$$

To find the value of x, substitute the value of y in either of the equations.

Thus from (1) we have $3x-35=13$;

$$\therefore\ x=\tfrac{48}{3}=16;$$

and from (2) we have $2x+49=81$;

$$\therefore\ x=\tfrac{32}{2}=16.$$

[PROOF. $48-35=13$, and $32+49=81$.]

Or the example may be solved thus:

$$3x-5y=13;\quad \therefore\ x=\frac{13+5y}{3};$$

$$2x+7y=81;\quad \therefore\ x=\frac{81-7y}{2};$$

$$\therefore\ \frac{13+5y}{3}=\frac{81-7y}{2},$$

that is, $26+10y=243-21y$;

$$\therefore\ 31y=217;$$

$$\therefore\ y=\tfrac{217}{31}=7; \text{ and so on.}$$

The first method is generally the more convenient.

Simultaneous equations of three unknown quantities are solved by eliminating one of the quantities by means of any pair of the equations, and then the *same* unknown quantity by means of another pair. In this way we reduce the three equations to two, each involving the same two unknown quantities, which can be solved by either of the methods of Ex. 1.

Ex. 2. *Given*

$$\left.\begin{array}{ll}(1) & 2x+3y+4z=16\\(2) & 3x+2y-5z=8\\(3) & 5x-6y+3z=6\end{array}\right\}$$

to find the values of x, y, z.

From (1) and (2)

$$\left.\begin{array}{r}6x+9y+12z=48\\ \underline{6x+4y-10z=16}\end{array}\right\}\ \begin{array}{l}\text{[(1) multiplied by 3.]}\\ \text{[(2) ,, ,, 2.]}\end{array}$$

By subtraction $5y+22z=32$ ……………………(4)

From (1) and (3) $\left.\begin{array}{r}10x+15y+20z=80\\ 10x-12y+\ \ 6z=12\end{array}\right\}$ [(1) multiplied by 5.]
[(3) ,, ,, 2.]

By subtraction $27y+14z=68$ ......(5)

We now have (4) $5y+22z=32$,

and (5) $27y+14z=68$;

$\therefore\ 135y+594z=864$ [(4) multiplied by 27.]

$135y+\ \ 70z=340$ [(5) ,, ,, 5.]

By subtraction $524z=524$

$\therefore\ z=1$,

and $5y+22=32$; (4)

$\therefore\ y=\frac{10}{5}=2$;

and $2x+6+4=16$; (1)

$\therefore\ x=\frac{6}{2}=3$.

EXAMPLES. X.

Solve the equations:

1. $3x+5y=8$, $4x+3y=7$.
2. $9x-4y=8$, $13x+7y=101$.
3. $4x-3y=2$, $5x-2y=9\frac{1}{2}$.
4. $x+y=17$, $2x-y=19$.
5. $4x-7y=26$, $4x+5y=50$.
6. $5x+\ y=32$, $3x-2y=14$.
7. $3x-7y=2$, $11y-3x=2$.
8. $3x+4y=11$, $15x-2y=11$.
9. $13x-6y=31$, $11x-3y=47$.
10. $x+y=6$, $x+z=8$, $y+z=10$.
11. $2x+y=9$, $2y+x=15$, $2z+y=27$.
12. $3x+5y=34$, $4x+2z=26$, $5y+\ z=32$.
13. $9x+10y=180$, $10y+11z=178$, $11z+12y=196$.
14. $x+y+z=15$, $x+z-y=5$, $x+y-z=-3$.
15. $x+y+z=36$, $5x-2y=10$, $9y-7x=110$.

Problems involving simultaneous equations.

Ex. 1. *Find two numbers such that twice the first added to three times the second is* 11, *and also such that four times the first added to the second is* 7.

Let x and y represent the numbers;

then (1) $2x+3y=11$,

and (2) $4x+\ y=7$,

that is, $4x+6y=22$ [(1) multiplied by 2.]

$4x+\ y=\ 7$

$\therefore$ by subtraction, $5y=15$

that is, $y=\frac{15}{5}=3$.

Substituting this value for y in the second equation, we have

$$\begin{aligned} 4x+3&=7; \\ \therefore\ 4x&=4; \\ \therefore\ x&=\tfrac{4}{4}=1. \end{aligned}$$

[PROOF. $2+9=11$, and $4+3=7$.]

Ex. 2. *Five times the sum of two numbers is* 70, *and the difference of the numbers is* 2: *what are the numbers?*

Let x and y represent the numbers;

$$\begin{aligned} &\text{then} & 5(x+y)&=70, \\ &\text{and} & x-y&=2; \\ &\text{hence} & 5x+5y&=70 \\ &\text{and} & \underline{5x-5y}&\underline{=10} && \text{[second equation multiplied by 5.]} \\ && \therefore\ 10y&=60 && \text{[by subtraction.]} \\ && \therefore\ y&=6; \\ &\text{and} & x-6&=2 && \text{[by second equation.]} \\ && \therefore\ x&=8. \end{aligned}$$

[PROOF. $5(8+6)=70$, and $8-6=2$.]

Ex. 3. *A number consisting of two digits is equal to* 9 *times the sum of the digits, and the digit in the tens' place is greater by* 7 *than the digit in the units' place: what is the number?*

It is to be observed that, when we represent algebraically numbers of more than one digit, we must remember that, say, 35 means, not 3×5, but $3\times 10+5$. Hence, if x is put for the tens' digit of a number and y for the units' digit, the number is represented algebraically by the expression $10x+y$.

Let x represent the tens' digit and y the units' digit;

$$\begin{aligned} \text{then}\quad (1)\quad 10x+y&=9(x+y), \\ (2)\quad x-y&=7. \end{aligned}$$

From (1) we have
$$\begin{aligned} 10x+y&=9x+9y; \\ \therefore\ x&=8y. \end{aligned}$$

From (2) we have
$$\begin{aligned} x-y&=7; \\ \therefore\ x&=7+y; \\ \therefore\ 8y&=7+y; \\ \therefore\ y&=1. \end{aligned}$$

From (2) we have
$$\begin{aligned} x-1&=7; \\ \therefore\ x&=8. \end{aligned}$$

Hence the required number is 81.

[PROOF. $10\times 8+1=9(8+1)$, and $8-1=7$.]

EXAMPLES. XI.

1. Find two numbers such that 3 times the first added to 4 times the second is 45, and also such that 5 times the first added to 2 times the second is 47.

2. The sum of two numbers is 32, and their difference 4: find the numbers.

3. 7 sheep and 3 oxen are worth £14, while, at the same prices, 6 sheep and 10 oxen are worth £38: find the value of a sheep and of an ox.

4. To one man I sell 5 horses and 7 cows for £124, and to another, at the same prices, 9 horses and 5 cows for £170: find what I get on the average for each horse and each cow.

5. 6 times the sum of two numbers is 330, and the difference of the numbers is 19: find the numbers.

6. The excess of one number over 30 is equal to the excess of 75 over another number, and the difference between the numbers is 65: what are the numbers?

7. The excess of one number over 25 is equal to the excess of 65 over another number, and the difference of the numbers is 16: find the numbers.

8. A number consisting of two digits is equal to 8 times the sum of its digits, and the digit in the units' place is less by 5 than the digit in the tens' place: find the number.

9. A farmer hired 8 men and 6 boys one day for £4. 16*s.* 0*d.*, and next day at the same rates he hired 6 men and 11 boys for £4. 18*s.* 0*d.*: how much per day did he pay to each?

10. I sold 6 horses and 7 cows for £107, and at the same prices 4 horses and 8 cows for £88: find how much I got for each.

CHAPTER III.

FACTORS, MEASURES, MULTIPLES, POWERS, ROOTS.

Factors, Measures and Multiples.

(*a*) $16 = 16 \times 1$. (*c*) $16 = 4 \times 4$.

(*b*) $16 = 8 \times 2$. (*d*) $16 = 4 \times 2 \times 2$.

(*e*) $16 = 2 \times 2 \times 2 \times 2$.

In each of these examples it is seen that the numbers equated with 16, when multiplied into each other, yield 16 as their product; they are therefore called *factors* of 16.

The factors of a number are the numbers which, when multiplied together, yield the number as their product.

It is obvious that every product is exactly divisible by each of its factors. Regarded from this point of view, a factor is called a *measure*, and the number into which it is divided is called a *multiple*. *Factors* are therefore correlated with *products*, and *measures* with *multiples*.

A number that has no factors but itself and 1 is called a *prime number;* hence in (*e*) the factors are *prime numbers* or *prime factors.*

A number that has other factors besides itself and 1 is called a *composite* number.

The *prime digits* are 1, 2, 3, 5, 7, and the *composite digits* are 4, 6, 8, 9. 2 is the only even number that is prime.

No number has more than one set of prime factors.

The student must discriminate between *prime numbers* and numbers that are *prime to each other.* 9 and 32 are *composite* numbers; for $9 = 3 \times 3$, and $32 = 2 \times 2 \times 2 \times 2 \times 2$, but they have no factor that is *common to both;* hence they are said to be *prime to each other.*

Show that a number is divisible:

(1) by 2 if its units' digit is even.

(2) by 4 if the number expressed by the last two digits is divisible by 4.

(3) by 8 if the number expressed by the last three digits is divisible by 8.

(4) by 3 if the sum of its digits is divisible by 3.

(5) by 6 if the units' digit is even, and the sum of the digits is divisible by 3.

(6) by 9 if the sum of the digits is divisible by 9.

(7) by 5 if the number ends in 5 or in 0.

(8) by 10 if the number ends in 0.

EXAMPLES. XII.

Find the prime factors of 64, 126, 84, 252, 108, 210, 432, 1872, 1184, 1140.

Common Factors.

Consider the following:

$$24 = 2 \times 2 \times 2 \times 3. \qquad 12 = 2 \times 2 \times 3. \qquad 18 = 2 \times 3 \times 3.$$

Observe here that one factor 2 and one factor 3 are common to the numbers 24, 12 and 18; they are therefore called *common factors* of these numbers; and, because 6, *the product of the common factors*, is the largest number that is contained exactly in all the numbers, it is called their *highest common factor* or their *greatest common measure.*

A number that is a factor of two or more numbers is said to be a *common factor* or a *common measure* of the numbers.

The *largest* number that will exactly divide each of two or more numbers is called the *highest common factor* or the *greatest common measure* of the numbers.

The letters H.C.F. are used to denote the *highest common factor*, and the letters G.C.M. to denote the *greatest common measure.*

NOTE. Unity or 1 is a common factor of all integral numbers, and is therefore disregarded in factorising.

Ex. 1. *Find the* H.C.F. *of* 32 *and* 36.

$$32 = 2 \times 2 \times 2 \times 2 \times 2. \qquad 36 = 2 \times 2 \times 3 \times 3.$$

The only prime factor common to 32 and 36 is here seen to be 2. In 32 there are 5 factors 2, and in 36 there are two factors 2; hence two factors 2 are common to 32 and 36; and no number greater than 4, the product of these two factors, is contained an exact number of times in 32 and 36;

∴ 4 is their H.C.F.

Ex. 2. *Find the* H.C.F. *of* 48, 72, *and* 96.

$$48 = 2 \times 2 \times 2 \times 2 \times 3. \qquad 72 = 2 \times 2 \times 2 \times 3 \times 3. \qquad 96 = 2 \times 2 \times 2 \times 2 \times 2 \times 3.$$

The prime factors common to 48, 72 and 96 are therefore three 2's and one 3; and no number greater than $2 \times 2 \times 2 \times 3$, or 24, the product of these factors, is contained an exact number of times in 48, 72 and 96;

∴ $\underline{24}$ is their H.C.F.

In a similar manner it may be shown *that in any series of numbers the product of their common prime factors is their* H.C.F. *or* G.C.M.

EXAMPLES. XIII.

Find the H.C.F. of:

1. 25, 75, and 100.
2. 16, 24, and 80.
3. 48, 64, and 120.
4. 35, 49, and 175.
5. 24, 48, 132, and 396.
6. 44, 48, 52, and 96.
7. 63, 700, and 371.
8. 84, 48, 132, and 396.
9. 126, 217, and 175.
10. 170, 262, and 568.

It is easy to see what factors are common to two or more small numbers, but it is sometimes very difficult to determine by inspection the factors of large numbers.

Highest Common Factor.

Let it be required to find the H.C.F. of 629 and 703, numbers that appear to be prime to each other. To prove whether they are or are not so, we operate upon them as follows:

$$\begin{array}{l}
629\,)\,703\,(\,1 \\
\underline{629} \\
74\,)\,629\,(\,8 \\
\underline{592} \\
37\,)\,74\,(\,2 \\
74
\end{array}$$

Or thus:

		Quotients.
629	703	1
74	629	8
37	74	2

The last divisor, namely, 37, is the H.C.F.

This method of finding the H.C.F. of two numbers is based upon the following principles:

(1) *A factor of a number is a factor of any multiple of the number.*

(2) *A common factor of two numbers is a factor of the difference of the numbers and of the difference of any multiples of the numbers.*

(3) *The largest factor of the divisor and the remainder is also the largest factor of the divisor and the dividend.*

In the working of the example, the student should observe:

(1) that the smaller number is divided into the larger;

(2) that, in the subsequent steps of the operation, the remainder is divided into the former divisor until there is no remainder;

(3) that the divisor that divides exactly into the last dividend is the H.C.F. of the numbers 629 and 703.

To prove that 37 is the H.C.F. of 629 and 703.

$74 = 37 \times 2$; $\therefore$ 37 is a factor of 74;

$629 = 592 + 37 = 74 \times 8 + 37 = 37 \times 2 \times 8 + 37 = 37 \times 16 + 37$
$= 37 \times 17$; $\therefore$ 37 is a factor of 629;

$703 = 629 + 74 = 37 \times 17 + 37 \times 2 = 37 \times 19$;
$\therefore$ 37 is a factor of 703;

$\therefore$ 37 is a common factor of 629 and 703;

and, since 17 and 19, the other factors, are prime to each other, no number greater than 37 is a factor of 629 and 703;

$\therefore$ 37 is the H.C.F. of 629 and 703.

Ex. 1. *Find the* H.C.F. *of* 948, 1185, *and* 1580.

948) 1185 (1
948
237) 948 (4
948

Or thus:

948	1185	1
237	948	4

$\therefore$ H.C.F. of 948 and 1185 is 237.

Now find the H.C.F. of 237 and 1580.

237) 1580 (6
1422
158) 237 (1
158
79) 158 (2
158

Or thus:

237	1580	6
158	237	1
79	158	2

$\therefore$ 79 is the H.C.F. of 237 and 1580.

Now, $948 = 79 \times 3 \times 4$. $1185 = 79 \times 5 \times 3$. $1580 = 79 \times 5 \times 4$.

Hence 79 is the only factor common to the three numbers;

$\therefore$ 79 is the H.C.F. of 948, 1185, and 1580.

Similarly we may determine the H.C.F. of several numbers by finding the H.C.F. of any two of them, then the H.C.F. of this factor and another of the numbers, then the H.C.F. of this result and another of the numbers, and so on.

Ex. 2. *Find the greatest number that will divide* 4091 *and* 8485, *and leave a remainder of* 9.

There is to be a remainder of 9;

∴ the dividends $= 4091 - 9$, or 4082, and $8485 - 9$, or 8476.

The greatest number that will divide 4082 and 8476 is their H.C.F.

$4082 = 26 \times 157$; $8476 = 26 \times 2 \times 163$;

∴ 26 is the required number.

Ex. 3. *Two tanks contain respectively* 7875 *and* 16128 *gallons: find the capacity of the largest vessel that can be filled an exact number of times by the contents of either tank.*

$7875 = 63 \times 125$; $16128 = 63 \times 256$.

The factors 125 and 256 being prime to each other, 63 is the H.C.F. of 7875 and 16128;

∴ a vessel containing 63 gallons is the largest that can be filled an exact number of times by the contents of each tank.

EXAMPLES. XIV.

Find the H.C.F. of:

1. 855 and 888.

2. 696 and 1305.

3. 232 and 3219.

4. 595 and 1071.

5. 949 and 1387.

6. 918, 1928, and 522.

7. 104811, 48825, and 68789.

8. 6916, 10686, and 14729.

9. 8214, 1110, and 1702.

10. 189, 252, 441, and 2205.

11. What is the greatest number that will divide 1578 and 14646, leaving in each case a remainder of 5?

12. What is the largest sum of money that can be paid an exact number of times out of £1. 5*s.* 10½*d.* and £2. 8*s.* 4¾*d.*?

13. Two fields of 24 and 28 acres respectively are to be divided into equal-sized paddocks as large as possible; find how many plots there will be and the size of each.

14. What is the length of the longest tape measure that can be used to measure exactly two cuttings 2916 ft. and 3582 ft. long respectively?

15. Two heaps of the same sized shot weigh respectively 10 cwt. 3 qrs. 15 lbs. and 5 cwt. 3 qrs. 23 lbs.: what is the greatest possible weight of each shot?

16. Two masses of gold, weighing 2542 grains and 5487 grains respectively, are to be made into medals of the same size: what is the greatest possible weight of each medal?

Powers and Roots.

$4 = 2 \times 2$; it is the product of two factors 2, and is therefore called the *second power* or the *square* of 2.

$8 = 2 \times 2 \times 2$; it is the product of three factors 2, and is therefore called the *third power* or the *cube* of 2.

$10000 = 10 \times 10 \times 10 \times 10$; it is the product of four factors 10, and is therefore called the *fourth power* of 10.

$a^5 = a \times a \times a \times a \times a$; it is the product of five factors a, and is therefore called the *fifth power* of a.

These results are often written thus:

$$\left.\begin{aligned} 4 &= 2^2 \\ 8 &= 2^3 \\ 10000 &= 10^4 \end{aligned}\right\{$$ the small *index* figure placed to the right and above the factor denoting how many times the factor is taken to produce the power.

The number raised to a power is called the *root* of the power. In the examples just given, 2 is the *square root* of 4, and the *cube root* of 8; 10 is the *fourth root* of 10000, and a the *fifth root* of a^5.

It is worth noticing in connection with the powers of 10 that the number of zeros in a power corresponds with the index figure of the root. Thus:

$$10^1 = 10;\ 10^2 = 100;\ 10^3 = 1000;\ 10^4 = 10000; \text{ and so on.}$$

The notation 2^2, 2^3, 10^4 is called *index notation*, which may be used with any number. Thus, if x represents a number, and n denotes the number of x's to be multiplied together, the product may be represented by x^n;

$$\text{for } x \times x \times x \times x \times x \ldots \text{ to } n \text{ factors} = x^n.$$

Observe that $5^3 \times 5^2 = 5 \times 5 \times 5 \times 5 \times 5 = 5^{3+2} = 5^5$;

that $$10^4 \times 10^3 = 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^{4+3} = 10^7;$$

that $$5^4 \div 5^2 = \frac{5 \times 5 \times 5 \times 5}{5 \times 5} = 5 \times 5 = 5^{4-2} = 5^2;$$

and that

$$10^5 \div 10^2 = \frac{10 \times 10 \times 10 \times 10 \times 10}{10 \times 10} = 10 \times 10 \times 10 = 10^{5-2} = 10^3.$$

These results are expressed by the following formulæ:

$$1.\ a^m \times a^n = a^{m+n}; \qquad 2.\ a^m \div a^n = a^{m-n};$$

where a, m, and n represent integers (positive), and (in the case of division) n is less than m.

$\sqrt{49}$, or $\sqrt[2]{49}$, or $49^{\frac{1}{2}}$ may be used to denote the square root of 49; $\sqrt{49}$ is the form generally used.

$\sqrt[3]{64}$, or $64^{\frac{1}{3}}$ may be used to denote the cube root of 64; $\sqrt[3]{64}$ is the form generally used, and a similar notation is used to denote higher roots.

The sign $\surd$ is said to be a corruption of the initial letter (r) of the word *radix*, the Latin word for *root*.

The figure put within the root sign denotes what root is to be extracted.

Our notation being founded on the *root* or *base* 10, it is easy to express a number in *index notation*, as:

$$\begin{aligned} 87359 &= 80000 + 7000 + 300 + 50 + 9 \\ &= 8 \times 10^4 + 7 \times 10^3 + 3 \times 10^2 + 5 \times 10^1 + 9. \end{aligned}$$

The following results should be carefully noted:

(*a*) $9^2 = (3 \times 3)^2 = 3^2 \times 3^2 = 81$;

$4^3 = (2 \times 2)^3 = 2^3 \times 2^3 = 64$.

Hence a power of a number = the product of the same power of its factors.

(*b*) $\sqrt{81} = \sqrt{9 \times 9} = \sqrt{9} \times \sqrt{9} = 3 \times 3 = 9$;

or $\sqrt{81} = \sqrt{3 \times 3 \times 3 \times 3} = \sqrt{3^2 \times 3^2} = \sqrt{3^2} \times \sqrt{3^2} = 3 \times 3 = 9$.

$\sqrt[3]{64} = \sqrt[3]{8 \times 8} = \sqrt[3]{8} \times \sqrt[3]{8} = 2 \times 2 = 4$;

or $\sqrt[3]{64} = \sqrt[3]{2 \times 2 \times 2 \times 2 \times 2 \times 2} = \sqrt[3]{2^3 \times 2^3}$

$= \sqrt[3]{2^3} \times \sqrt[3]{2^3} = 2 \times 2 = 4$.

Hence a root of a number = the product of the same root of its factors.

As we have seen, the product of *literal* factors is usually denoted by writing the factors side by side without any sign between them, as:

abc, which means $a \times b \times c$;

$a^2b^2c^2$, which means $a \times a \times b \times b \times c \times c$.

EXAMPLES. XV.

(*a*) Find the values of

1. $3^2 \times 3^2$; $5^2 \times 5$; $(3 \times 2)^4$; $4^7 \div 4^2$; $(7 \times 7)^4 \div 7^4$.

2. $(2 \times 2 \times 3)^3$; 13^3; $15^6 \div 15^3$; $10^{10} \div 10^4$; $20^6 \div 20^4$.

3. $\sqrt{121}$; $\sqrt{256}$; $\sqrt{625}$; $\sqrt{4096}$; $\sqrt{1225}$; $\sqrt{729}$.

4. $\sqrt[3]{125}$; $\sqrt[3]{216}$; $\sqrt[3]{512}$; $\sqrt[3]{1728}$; $\sqrt[3]{1331}$.

5. 45×10^2; 27×10^5; $350000 \div 10^4$; $100^4 \div 100^3$.

(*b*) If $a=3$, $b=5$, $c=7$, and $d=0$, find the values of :

1. ab^2c; $(ab)^2$; $(ad)^2$; $3ab^2-2c^2$; $\frac{7c+2b^2}{11}$.

2. $\frac{5a^3}{a^2}$; $\frac{a^3b^3c^3}{a^2b^2c^2}-ab$; $5ad+b^2c^2-5ab^2+a^4$.

3. $\sqrt{a^2b^2c^2}$; $\sqrt[3]{(bc)^3}$; $\sqrt[3]{(a+b)}$; $\sqrt{(b+c+4)}$; $\frac{(bc)^2-(ab)^2}{(ac)^2}$.

(*c*) Express in index notation :

85763 ; 375683 ; one million five hundred thousand.

Any number that has an exact square root is called a square number. Thus 9, 25, 36, 81, 144 are square numbers, for the square root of 9 is 3, of 25 is 5, of 36 is 6, of 81 is 9, of 144 is 12.

In practical work squares and square roots are concerned chiefly with the mensuration of lines and surfaces. You have learnt in the elements of arithmetic that, *when the number of linear units in the length is multiplied by the number of linear units in the breadth of a rectangular surface, the product gives the number of square units of the same denomination in the area of the rectangle.* A square is a rectangle the sides of which are equal; hence, *if we square the number of units in the side of a square, we get the number of square units of the same denomination in the area of the square.*

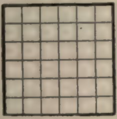


FIG. 1.

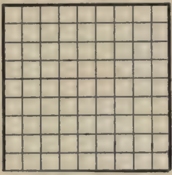


FIG. 2.

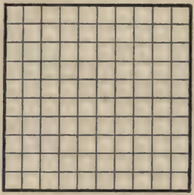


FIG. 3.

NOTE. Any four-sided figure that has all its angles right angles is a rectangle.

Fig. 1 is a square of side 6 units, and is seen to contain 36, *i.e.* (6^2) square units; Fig. 2 is a square of side 9 units, and is seen to contain 81, *i.e.* (9^2) square units; and Fig. 3 is a square of side 10 units, and is seen to contain 100, *i.e.* (10^2) square units.

$$\text{Hence area of square} = (\text{side})^2,$$

from which it follows that

$\sqrt{\text{number of square units in a square}}$ = number of linear units in side of the square.

Fig. 1 contains 36 square units;

$$\therefore \text{ length of side} = \sqrt{36} = 6 \text{ units.}$$

Fig. 2 contains 81 square units;

$$\therefore \text{ length of side} = \sqrt{81} = 9 \text{ units.}$$

Fig. 3 contains 100 square units;

$$\therefore \text{ length of side} = \sqrt{100} = 10 \text{ units.}$$

These results may be expressed generally as follows:

$$\begin{aligned} &\text{If } l = \text{side of square,} \\ &\text{then area of square} = l^2, \\ &\text{and side of square} = \sqrt{l^2}. \end{aligned}$$

In most cases the square root of a square number is easily found by the method of factorising. This has been already exemplified by small numbers; we shall now show how to apply the method to large numbers.

Ex. *The area of square garden is 2704 square yards: find the length of the fence that encloses it.*

$$\begin{aligned} \sqrt{2704} &= \sqrt{2 \times 2 \times 2 \times 2 \times 13 \times 13} \\ &= \sqrt{2^2 \times 2^2 \times 13^2} \\ &= 2 \times 2 \times 13 \\ &= 52 \text{ yards} = \text{length of side;} \\ \therefore \text{ perimeter} &= (52 \times 4) \text{ yards} \\ &= 208 \text{ yards} \\ &= \text{length of enclosing fence.} \end{aligned}$$

$$\begin{array}{r|r} 2 & 2704 \\ \hline 2 & 1352 \\ \hline 2 & 676 \\ \hline 2 & 338 \\ \hline 13 & 169 \\ \hline & 13 \end{array}$$

EXAMPLES. XVI.

1. The area of a square plot is 1296 square yards: what is the length of the fence that surrounds it?

2. The area of a square field is 5625 metres: find the cost of fencing it at 2*s*. 6*d*. per metre.

3. The area of a square field is 2 acres 1984 square yards: find the perimeter of the field.

4. A square estate contains 9801 square chains: find the cost of fencing it at £2. 10*s*. a chain.

The following results are sometimes useful:

(1) $$(a+b)^2=(a+b)(a+b).$$

To multiply $(a+b)$ by $(a+b)$ is to multiply each term of the multiplicand by each term of the multiplier and sum the results;

$$\begin{aligned} \therefore\ (a+b)(a+b) &= a(a+b)+b(a+b) \\ &= a^2+ab+ab+b^2 \\ &= a^2+2ab+b^2. \end{aligned}$$

That is, *the square of the sum of two numbers is equal to the sum of the squares of the numbers together with twice the product of the numbers.*

Ex.
$$\begin{aligned} (105)^2 &= (100+5)^2 \\ &= (100)^2+100\times5\times2+5^2 \\ &= 10000+1000+25 \\ &= 11025. \end{aligned}$$

(2)
$$\begin{aligned} (a-b)^2 &= (a-b)(a-b) \\ &= a(a-b)-b(a-b) \\ &= a^2-ab-ab+b^2 \\ &= a^2-2ab+b^2. \end{aligned}$$

That is, *the square of the difference of two numbers is equal to the sum of the squares of the numbers less twice the product of the numbers.*

Ex.
$$\begin{aligned} (126)^2 &= (130-4)^2 \\ &= (130)^2-130\times4\times2+4^2 \\ &= 16900-1040+16 \\ &= 15876. \end{aligned}$$

(3)
$$\begin{aligned} (a+b)(a-b) &= a(a+b)-b(a+b) \\ &= a^2+ab-ab-b^2 \\ &= a^2-b^2. \end{aligned}$$

That is, *the difference of the squares of two numbers is equal to the product of the sum and difference of the numbers.*

Exs.
$$\begin{aligned} (12)^2-5^2 &= (12+5)(12-5) \\ &= 17\times7=119. \end{aligned}$$

$$\begin{aligned} (29)^2-(25)^2 &= (29+25)(29-25) \\ &= 54\times4=216. \end{aligned}$$

EXAMPLES. XVII.

Write down the values of :

1. $(2x+3y)^2$, $(5x+8y)^2$, $(9x+3y)^2$.
2. $(2x-3y)^2$, $(5x-8y)^2$, $(9x-3y)^2$.
3. $(2a+3b)(2a-3b)$, $(3x+5y)(3x-5y)$.
4. $(3x-4y)(3x+4y)$, $(2x^2+4y^2)(2x^2-4y^2)$.
5. $(7x-5y)^2$, $(7x+5y)^2$, $(6x^2-3y^2)^2$.
6. $(49)^2$, $(78)^2$, $(209)^2$, $(325)^2$.
7. $(298)^2$, $(491)^2$, $(588)^2$, $(997)^2$.
8. $(49)^2-(31)^2$, $(96)^2-(84)^2$, $(125)^2-(105)^2$, $(899)^2-(799)^2$.

Least Common Multiple.

As we have seen, a number is called a *multiple* of another when it contains the other an exact number of times.

A number is called a *common multiple* of two or more others when it is a *multiple* of each of them.

Thus, 36 contains an exact number of twos, of threes, of fours, of sixes, of nines, of twelves, and of eighteens; 36 is therefore a *common multiple* of 2, 3, 4, 6, 9, 12, and 18.

It is obvious that the *product* of two or more numbers is a *common multiple* of the numbers.

Thus, abc is a multiple of a, b, and c; for a is contained in it bc times, b ac times, and c ab times.

The *least common multiple* of two or more numbers is the *smallest* number that contains each of the numbers.

The *least common multiple* is denoted by the letters L.C.M.

Ex. 1. *To find the* L.C.M. *of two numbers, say* 18 *and* 24.

$$18 = 2\times3\times3 \quad = 2\times3^2;$$
$$24 = 2\times2\times2\times3 = 2^3\times3.$$

The only different prime factors of 18 and 24 are 2 and 3, and therefore any multiple of 18 and 24 must contain these factors, and the least multiple of them must contain these prime factors only.

Since 18 contains the factor 3^2 and 24 the factor 2^3, no number that does not contain the factors 3^2 and 2^3 can contain 18 and 24. The smallest number that contains 3^2 and 2^3 is the product of 3^2 and 2^3, that is, 72; and upon trial it will be found that 72 is the L.C.M. of 18 and 24.

Notice that the same result is gained by dividing the product of the two numbers by their H.C.F. (G.C.M.). Thus:

$$\frac{24 \times 18}{6} = 72,$$

from which are derived the following results:

H.C.F. × L.C.M. of two numbers = the product of the numbers; (1)

$$\text{H.C.F. of two numbers} = \frac{\text{product of the numbers}}{\text{L.C.M. of the numbers}}; \quad \ldots\ldots\ldots(2)$$

$$\text{L.C.M. of two numbers} = \frac{\text{product of the numbers}}{\text{H.C.F. of the numbers}}. \quad \ldots\ldots\ldots(3)$$

General proof:

Let a and b represent the two numbers, and c their H.C.F. or G.C.M.; and let $a = cx$, and $b = cy$, x and y being different prime factors.

Because $a = cx$, and $b = cy$,

$$\therefore \ ab = cx \times cy = c \times cxy;$$

but c is the H.C.F. of a and b;

$\therefore$ cxy is the L.C.M. of a and b,

and $c \times cxy = ab$. ....(1)

Since $c \times cxy = ab$;

$$\therefore \ c = \frac{ab}{cxy}, \quad \ldots\ldots\ldots\ldots(2)$$

$$\text{and} \quad cxy = \frac{ab}{c}. \quad \ldots\ldots\ldots\ldots(3)$$

Ex. 2. *To find the* L.C.M. *of more than two numbers*, 54, 81, 63, *and* 14.

$$54 = 2 \times 3 \times 3 \times 3 = 2 \times 3^3,$$
$$81 = 3 \times 3 \times 3 \times 3 = 3^4,$$
$$63 = 3 \times 3 \times 7 \quad = 3^2 \times 7,$$
$$14 = 2 \times 7.$$

The different prime factors are 2, 3, and 7, and therefore the L.C.M. must contain these and these alone; but it is clear that it must contain the highest power of each that occurs in the numbers 54, 81, 63, and 14.

Hence the L.C.M. $2 \times 3^4 \times 7 = 1134$.

And always *the* L.C.M. *of two or more numbers may be found by taking the product of the highest powers of their different prime factors.*

Ex. 3. *A can dig 14 yards of a ditch in a day, B 18 yards, C 22 yards, and D 24 yards: find the least number of yards that would give an exact number of days' work to each of them.*

The smallest number of yards that will contain the number of yards done by each daily is evidently the required number.

$$14 = 2 \times 7. \qquad 22 = 2 \times 11.$$
$$18 = 2 \times 3 \times 3. \qquad 24 = 2 \times 2 \times 2 \times 3.$$
$$\therefore \text{ L.C.M.} = 2^3 \times 3^2 \times 7 \times 11 = 5544;$$
$$\therefore \text{ required yards of ditch} = \underline{5544}.$$

Ex. 4. *Four men start at the same place and at the same time to walk round a garden: A can go round in 9 minutes, B in 10 minutes, C in 12 minutes, and D in 15 minutes: in what time will they be again at the starting point together?*

The least number of minutes that contains 9, 10, 12, and 15 exactly is the time.

$$9 = 3 \times 3. \qquad 12 = 2 \times 2 \times 3.$$
$$10 = 2 \times 5. \qquad 15 = 3 \times 5.$$
$$\therefore \text{ L.C.M.} = 2^2 \times 3^2 \times 5 = 180;$$

$\therefore$ they are at the starting point together in $\underline{\text{180 minutes,}}$ *A* having gone 20 rounds, *B* 18 rounds, *C* 15 rounds, and *D* 12 rounds.

EXAMPLES. XVIII.

Find the L.C.M. of:

1. 28, 56, 84.

2. 17, 51, 34.

3. 2, 8, 15, 24.

4. 5, 15, 25, 55.

5. 8, 12, 15, 20, 120.

6. 9, 24, 36, 57.

7. 17, 50, 85, 90.

8. 7, 10, 11, 22, 35.

9. 3, 12, 28, 27, 4, 6.

10. 26, 13, 5, 40, 15.

11. 8, 10, 12, 14, 16, 20, 24, 28.

12. 84, 96, 68, 48.

13. What is the shortest length of a trench that can be measured exactly with tapes 54, 216, and 168 inches long respectively?

14. The books in a case can be arranged in rows of 5, 7, 9, or 15: how many are there?

15. What is the least number of persons that can be divided into groups of 16, 30, 48, 56, and 72?

16. What is the least number which, when divided by 12, 20, 30, or 54, leaves a remainder of 4?

17. Find the number of miles in the radius of the earth, having given that it is the least number that is divisible by 2, 3, 4, 5, 6, 8, 9, 10, 11, and 12.

18. Four bells toll at intervals of 3, 7, 12, and 14 seconds respectively, and begin to toll at the same instant: find when they will next toll together.

CHAPTER IV.

VULGAR FRACTIONS AND RATIO.

A FRACTION is any part of a whole. In mathematics it is defined as one or more *equal* parts of a whole.

Any whole may be regarded as a unit, and be denoted by 1 or unity. Thus the accompanying figure is a unit, 1 *square inch.*

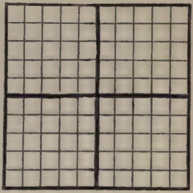


FIG. 4.

The figure is seen to be divided by the heavy lines into four equal parts, each being $\frac{1}{4}$ (1 fourth) of the unit; the unit is therefore divided into fourths or measured by fourths, which are notated thus: $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$, read 1 fourth, 2 fourths, 3 fourths, 4 fourths. It is obvious that the expression $\frac{4}{4}$ comprises the whole unit. In reality it is not a fraction at all, for it denotes a whole unit. It should be noted, however, that it denotes the whole in *terms of fourths*, and therefore does not mean quite the same thing as 1, which denotes an *undivided whole.*

Note that the lower number of the fraction, besides denoting the number of equal parts into which the unit is divided, gives a *name* to the parts, and that the upper number denotes the number of equal parts included in the fraction. Hence the lower number is called the *Denominator* and the upper number the *Numerator*; they are also called the *Terms* of the fraction.

If the numerator is less than the denominator, the fraction is called a *proper fraction.* A fractional form that denotes unity or a quantity greater than unity is called an *improper fraction.* Thus, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$ are proper fractions, and $\frac{4}{4}$, $\frac{9}{8}$, $\frac{12}{9}$ are improper fractions.

Notice that each of the fourths into which the unit is divided by the heavy lines contains 25 of the equal parts that make up the whole figure, an examination of which shows that, in each of the values $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$, there are 25 times as many hundredths as fourths. Hence:

(1) $\frac{1}{4}$ of the unit $=\frac{1\times25}{4\times25}=\frac{25}{100}$ of the unit,

that is, $\frac{1}{4}$ of a unit $=\frac{25}{100}$ of the same unit.

(2) $\frac{2}{4}$ of the unit $=\frac{2\times25}{4\times25}=\frac{50}{100}$ of the unit,

that is, $\frac{2}{4}$ of a unit $=\frac{50}{100}$ of the same unit.

(3) $\frac{3}{4}$ of the unit $=\frac{3\times25}{4\times25}=\frac{75}{100}$ of the unit,

that is, $\frac{3}{4}$ of a unit $=\frac{75}{100}$ of the same unit.

(4) $\frac{4}{4}$ of the unit $=\frac{4\times25}{4\times25}=\frac{100}{100}$ of the unit,

that is, $\frac{4}{4}$ of a unit $=\frac{100}{100}$ of the same unit.

And always $\frac{a}{b}=\frac{ac}{bc}$.

Hence: *Multiplying the terms of a fraction by the same number does not alter the value of the fraction.*

It is obvious from the figure that the larger the number of equal parts into which unity is divided, the smaller is each such part.

EXAMPLES. XIX.

(*a*) Show from the figure that:

$$\frac{5}{100}=\frac{1}{20};\quad \frac{10}{100}=\frac{1}{10};\quad \frac{30}{100}=\frac{3}{10};\quad \frac{15}{100}=\frac{3}{20};$$

$$\frac{45}{100}=\frac{9}{20};\quad \frac{7}{20}=\frac{35}{100};\quad \frac{19}{20}=\frac{95}{100};\quad \frac{14}{20}=\frac{70}{100}.$$

(*b*) Write down the number for which x stands in each of the following equations:

$$\frac{7}{20}=\frac{x}{100};\quad \frac{4}{25}=\frac{x}{100};\quad \frac{17}{20}=\frac{x}{60};$$

$$\frac{1}{2}=\frac{x}{8};\quad \frac{9}{14}=\frac{x}{56};\quad \frac{x}{48}=\frac{7}{12}.$$

Fractions, the denominators of which are a power of 10, are called *decimal fractions.* All others are called *vulgar fractions* (*i.e. common fractions*).

In decimal fractions the denominator is generally not expressed. These fractions have a notation of their own, which is an extension of integral notation. Thus, $\frac{25}{100}$ is written ·25, $\frac{75}{100}$ is written ·75, and so on, the point · showing where the decimal parts begin.

Vulgar fractions take the form $\frac{a}{b}$ or a/b, which means that unity is divided into b equal parts and that a of these parts are taken, that is, $\frac{a}{b}=a$ times $\frac{1}{b}$ of the unit.

Besides denoting a times $\frac{1}{b}$ of a unit, $\frac{a}{b}$ is used to denote the bth part of a, and also the number of times the quantity denoted by a contains the quantity denoted by b, that is, the *quantitative relation* of quantity a to quantity b. The mathematical word for relation in this sense is *ratio.*

Hence the expression $\frac{a}{b}$ may denote:

(1) a times $\frac{1}{b}$ of a unit;

(2) the bth part of a;

(3) the quotient of a divided by b,

(4) the *quantitative relation* or *ratio* of a to b.

We have already seen that

$$\frac{1}{4}=\frac{1\times 25}{4\times 25}=\frac{25}{100},$$

a result that may now be read:

1 has the same ratio to 4 that 25 has to 100, or, more briefly, 1 is to 4 as 25 is to 100, or 1 : 4 : : 25 : 100.

Also 4 is to 1 as 100 is to 25,

$$\text{for}\quad \frac{4}{1}=\frac{100}{25}.$$

Graphical Representation of Ratio.

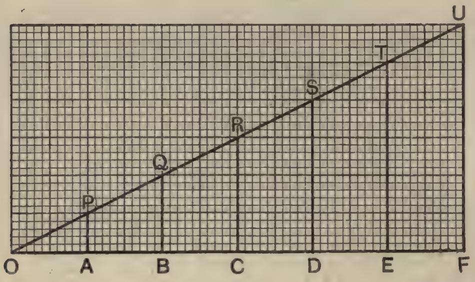


FIG. 5.

$$\frac{PA}{OA}=\frac{1}{2}.$$

$$\frac{QB}{OB}=\frac{2}{4}=\frac{1}{2}.$$

$$\frac{RC}{OC}=\frac{3}{6}=\frac{1}{2}.$$

$$\frac{SD}{OD}=\frac{4}{8}=\frac{1}{2}.$$

$$\frac{TE}{OE}=\frac{5}{10}=\frac{1}{2}.$$

$$\frac{UF}{OF}=\frac{6}{12}=\frac{1}{2}.$$

It is obvious that $\frac{25}{100}$, $\frac{50}{100}$, $\frac{75}{100}$, $\frac{100}{100}$ of the unit may, by a reversal of the process, be expressed in fourths of the unit.

Thus:

(1) $\frac{25}{100}$ of the unit $=\frac{25\div 25}{100\div 25}=\frac{1}{4}$ of the unit.

(2) $\frac{50}{100}$ of the unit $=\frac{50\div 25}{100\div 25}=\frac{2}{4}$ of the unit.

(3) $\frac{75}{100}$ of the unit $=\frac{75\div 25}{100\div 25}=\frac{3}{4}$ of the unit.

(4) $\frac{100}{100}$ of the unit $=\frac{100\div 25}{100\div 25}=\frac{4}{4}$ of the unit.

And always $\frac{a}{b}=\frac{a\div c}{b\div c}$, where c is a factor of a and b.

Hence: *Dividing the terms of a fraction by the same number does not alter the value of the fraction.*

From this it follows that, when the terms of a fraction have a common factor, the common factor may be cancelled without altering the value of the fraction.

When the terms of a fraction are *prime* to each other, the fraction is said to be in its lowest terms. All fractional answers must be so expressed.

Ex. 1. *Express* $\frac{27}{36}$ *in its lowest terms.*

$$\frac{27}{36}=\frac{3\times3\times3}{3\times3\times4}=\frac{3^2\times3}{3^2\times4}.$$

Dividing the terms of the fraction by their highest common factor 3^2, we have $\frac{3}{4}$, the terms of which are prime to each other.

The process of eliminating factors that are common to the terms of a fraction is called *cancelling.*

Examination of the accompanying figure, which is divided by the heavy lines into fourths and by the light lines in thirty-sixths, shows that $\frac{3}{4}$ and $\frac{27}{36}$ represent the same fraction of the whole figure, for each part of the first comprises 9 parts of the second. Each thirty-sixth being $\frac{1}{9}$ of the fourth, there are 9 times as many thirty-sixths as fourths.

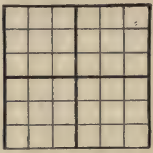


FIG. 6.

Show from the figure that $\frac{27}{36}=\frac{9}{12}=\frac{3}{4}$.

These results show that a fraction is reduced to its lowest terms when its numerator and denominator are divided by all the factors common to both. It is obvious that dividing numerator and denominator by all the factors common to both is the same in effect as dividing them by their highest common factor, for the H.C.F. is the product of all the common factors.

Ex. 2. *Reduce* $\frac{589}{775}$ *to its lowest terms.*

The terms of this fraction seem to be prime to each other. Testing by the rule for finding the H.C.F., we find that the factor 31 is common to both. Hence

$$\frac{589}{775}=\frac{19\times31}{25\times31}=\frac{19}{25}.$$

Ex. 3. *Express* $\frac{4a^2b}{6ab^2}$ *in its simplest form.*

$$\frac{4a^2b}{6ab^2}=\frac{2\times2\times a\times a\times b}{2\times3\times a\times b\times b}=\frac{2ab\times2a}{2ab\times3b}=\frac{2a}{3b}.$$

The expression $2ab$, being common to numerator and denominator, is cancelled out.

EXAMPLES. XX.

Reduce the following fractions to their lowest terms:

1. $\frac{51}{85}$.
2. $\frac{77}{143}$.
3. $\frac{117}{156}$.
4. $\frac{119}{272}$.
5. $\frac{270}{522}$.
6. $\frac{299}{851}$.
7. $\frac{589}{1736}$.
8. $\frac{1161}{1290}$.
9. $\frac{696}{1305}$.
10. $\frac{856}{936}$.
11. $\frac{595}{1071}$.
12. $\frac{949}{1387}$.
13. $\frac{1081}{1311}$.
14. $\frac{1850}{3003}$.
15. $\frac{2108}{3813}$.
16. $\frac{2772}{6468}$.
17. $\frac{1547}{5712}$.
18. $\frac{4844}{5536}$.
19. $\frac{a^2b}{ab^3}$.
20. $\frac{3x^2y^3}{6x^3y}$.
21. $\frac{9mn^3}{12m^2n}$.
22. $\frac{15pq^3}{20p^3q^2}$.
23. $\frac{a^2b^3c^4}{a^2b^2c^2}$.
24. $\frac{x^5y^3z^2}{x^3y^2z^2}$.

A number that consists of the sum of a whole number and a fraction is called a *mixed number*, as, for example, $3 + \frac{7}{25}$, which, in the notation of fractions, is written $3\frac{7}{25}$.

Since any whole number may be regarded as a fraction with 1 for its denominator, $3 = \frac{3}{1}$. Hence,

$$3\tfrac{7}{25} = \tfrac{3}{1} + \tfrac{7}{25} = \tfrac{3 \times 25}{25} + \tfrac{7}{25} = \tfrac{75}{25} + \tfrac{7}{25} = \tfrac{82}{25},$$

read 82 twenty-fifths).

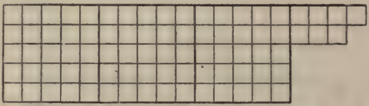


FIG. 7.

We have here expressed a mixed number as an improper fraction.

We may represent the work graphically. Consider each of the large squares in the accompanying figure as a unit; then the whole figure is represented numerically by $3 + \frac{7}{25}$ or $3\frac{7}{25}$, for each of the 7 small squares is seen to be $\frac{1}{25}$ of a unit; and, since there are in each unit 25 of the small squares, it is clear that in the three units there are 3 times 25 small squares. Hence the total number of small squares or *twenty-fifths* in the whole figure is 75 + 7 or 82. It is obvious that the 82 twenty-fifths may be expressed as a mixed number by dividing 82 by 25, thus:

$$\tfrac{82}{25} = 3\tfrac{7}{25}.$$

EXAMPLES. XXI.

Convert the following mixed numbers into improper fractions:

1. $9\frac{1}{8}$. 3. $32\frac{7}{12}$. 5. $21\frac{16}{25}$. 7. $41\frac{19}{20}$. 9. $390\frac{10}{19}$.
2. $27\frac{5}{6}$. 4. $17\frac{23}{24}$. 6. $30\frac{1}{15}$. 8. $473\frac{17}{18}$. 10. $570\frac{16}{27}$.

EXAMPLES. XXII.

Convert the following improper fractions into mixed numbers:

1. $\frac{12}{5}$. 3. $\frac{25}{9}$. 5. $\frac{79}{15}$. 7. $\frac{256}{12}$. 9. $\frac{540}{5}$.
2. $\frac{17}{8}$. 4. $\frac{39}{16}$. 6. $\frac{101}{7}$. 8. $\frac{765}{16}$. 10. $\frac{791}{19}$.

As we have seen, we can convert a fraction into an equivalent fraction by multiplying each of its terms by the same number. Thus:

$$\frac{2}{3} = \frac{2 \times 2}{3 \times 2} = \frac{4}{6} = \frac{4 \times 7}{6 \times 7} = \frac{28}{42}.$$

In other words, 2 has the same relation or *ratio* to 3 that 28 has to 42.

[Prove by using squared paper.]

In the same way we can express a fraction (or fractional form) as an equivalent fraction (or fractional form) with any denominator that is a *multiple* of the denominator of the given fraction (or fractional form).

Ex. 1. *Express* 19 *as a fraction with a denominator of* 8.

$$19 = \frac{19}{1} = \frac{19 \times 8}{1 \times 8} = \frac{152}{8};$$

$$\therefore \frac{152}{8} \text{ is the required fraction.}$$

19 contains 1 as many times as 152 contains 8; hence the ratio of 19 to 1 = the ratio of 152 to 8.

[Prove by using squared paper.]

Note. $\frac{19}{1}$ and $\frac{152}{8}$ are fractions only in form, for they denote an exact number of wholes or units.

Ex. 2. *Express $\frac{7}{8}$ as an equivalent fraction with a denominator of* 72.

To raise the denominator 8 to 72 we must multiply it by 9; and, if we multiply the denominator by 9, we must also multiply the numerator by 9.

$$\frac{7}{8}=\frac{7\times9}{8\times9}=\frac{63}{72};$$

$$\therefore\ \frac{63}{72} \text{ is the required fraction.}$$

That is, the ratio $\frac{7}{8}$ = the ratio $\frac{63}{72}$.

[Prove by using squared paper.]

The same example worked by the methods of algebra.

Let x represent the numerator of the fraction the denominator of which is 72.

Then
$$\frac{x}{72}=\frac{7}{8};$$

$$\therefore\ x=\frac{7\times72}{8}\quad\text{(both sides}\times72)$$

$$=63;$$

$$\therefore\ \frac{63}{72} \text{ is the required fraction.}$$

Ex. 3. *Find the number that has the same ratio to* 63 *that* 3 *has to* 7.

Let x represent the number, then

$$\frac{x}{63}=\frac{3}{7};$$

$$\therefore\ 7x=63\times3;\quad\text{(both sides}\times63)$$

$$\therefore\ x=\frac{63\times3}{7}=27.$$

EXAMPLES. XXIII.

1. Find fractions with denominators 168 which are respectively equal to $\frac{5}{8}$, $\frac{7}{12}$, $\frac{9}{14}$, $\frac{25}{28}$.

2. Find the value of x in each of the following pairs of equal ratios:

$$\frac{x}{35}=\frac{2}{7};\qquad \frac{x}{144}=\frac{7}{12};\qquad \frac{x}{165}=\frac{14}{15};$$

$$\frac{13}{28}=\frac{x}{420};\qquad \frac{6}{x}=\frac{25}{75};\qquad \frac{13}{91}=\frac{14}{x}.$$

3. Express (in lowest terms) 27 as a fraction of 135; 19 as a fraction of 171; 9 as a fraction of 78, ab^2 as a fraction of bc.

4. (1) What part of a cake worth 8*s.* 2*d.* can be bought with 2*s.* 4*d.*?

(2) What part of a plank worth 8*s.* 3*d.* can be bought with 1*s.* 10*d.*?

5. (1) Find the number that has the same ratio to 9 that 12 has to 96.

(2) Find the number that has the same ratio to 25 that 1 has to 2.

The Comparison of Concrete Quantities.

A ratio denotes the relation of two quantities *of the same kind*; for example, of a sum of money to another sum of money, of a length to a length, and so on, not of money to length, nor of money to an abstract number, and so on. Moreover, the numerator and denominator of the fraction expressing the ratio must denote the *same kind of unit* of the quantities compared.

Suppose it is required to find the ratio of 3*s.* 6*d. to* 7*s.* 6*d.*

$$\frac{3s.\ 6d.}{7s.\ 6d.}=\frac{7 \text{ sixpences}}{15 \text{ sixpences}}\quad\left[\text{both quantities expressed in terms of sixpences.}\right]$$

$$=\tfrac{7}{15}\ [\text{both quantities now regarded as abstract numbers}];$$

$\therefore$ the ratio of 3*s.* 6*d.* to 7*s.* 6*d.* is as 7 to 15,

that is, 3*s.* 6*d.* is $\frac{7}{15}$ of 7*s.* 6*d.*

EXAMPLES. XXIV.

1. Find the ratio of 2*s.* 6*d.* to 7*s.* 6*d.*, of £1. 10*s.* to £10. 10*s.*, of 4*s.* 9*d.* to 10*s.* 1*d.*, of 15*s.* to 6*s.* 3*d.*, of 3 feet 6 inches to 8 yards 6 inches, of x yards to y inches, of c cwt. to d pounds.

2. Draw a line x inches long, and on it mark off y inches and z inches (the sum of y and z to be less than x): what fraction of the whole line now remains?

3. Find the amount of money that has the same ratio to 2*s.* 6*d.* that 1 foot 3 inches has to 10 inches.

4. The ratio of 7*s.* 6*d.* to the money in my purse is equal to the ratio of 1 cwt. 3 qrs. to 3 cwt. 2 qrs.: find the amount in the purse.

The conversion of two or more fractions into fractions with a common denominator.

Ex. 1. *Convert* $\frac{3}{7}$, $\frac{1}{4}$, *and* $\frac{9}{14}$ *into equivalent fractions with a common denominator.*

Here the fractions are to have the same denominator. Hence the denominator must be a multiple of each of the three denominators, that is, a *common multiple* of 7, 4, and 14. Take the L.C.M.

The L.C.M. of 7, 4, and 14 is 28. Now, to produce 28 the denominators must be multiplied by 4, 7, and 2 respectively.

$$\frac{3}{7}=\frac{3\times4}{7\times4}=\frac{12}{28},\quad \frac{1}{4}=\frac{1\times7}{4\times7}=\frac{7}{28},\quad \frac{9}{14}=\frac{9\times2}{14\times2}=\frac{18}{28};$$

$\therefore$ $\frac{12}{28}$, $\frac{7}{28}$, $\frac{18}{28}$ are fractions equivalent to $\frac{3}{7}$, $\frac{1}{4}$, and $\frac{9}{14}$ respectively, and having the common denominator 28; and, since 28 is the smallest denominator common to the fractions $\frac{3}{7}$, $\frac{1}{4}$, $\frac{9}{14}$, it is called the **least common denominator** (L.C.D.) of these fractions. But the question says a common denominator; hence we may take any multiple of 28 as the common denominator. Take 56.

Then $$\frac{3}{7}=\frac{3\times 8}{7\times 8}=\frac{24}{\underline{56}}, \qquad \frac{1}{4}=\frac{1\times 14}{4\times 14}=\frac{14}{\underline{56}},$$

$$\frac{9}{14}=\frac{9\times 4}{14\times 4}=\frac{36}{\underline{56}}; \text{ and so on.}$$

When fractions of a unit or of equal units of the same kind are expressed in terms of a common denominator, they can be compared, or added, or subtracted, for the equal parts denoted by a common denominator are of the same magnitude.

Comparison: Inspection of $\frac{3}{7}$, $\frac{1}{4}$, and $\frac{9}{14}$, when they are expressed in their equivalent forms $\frac{12}{28}$, $\frac{7}{28}$, and $\frac{18}{28}$, shows that $\frac{9}{14}$ is the largest and $\frac{1}{4}$ the smallest of the fractions. Arranged in ascending order of magnitude, the fractions are seen to be $\frac{7}{28}$, $\frac{12}{28}$, $\frac{18}{28}$; arranged in descending order, they are $\frac{18}{28}$, $\frac{12}{28}$, $\frac{7}{28}$.

Addition: $\frac{3}{7}+\frac{1}{4}+\frac{9}{14}=\frac{12}{28}+\frac{7}{28}+\frac{18}{28}=\frac{37}{28}=1\frac{9}{28}$.

The addition of $\frac{12}{28}$, $\frac{7}{28}$, and $\frac{18}{28}$ is the same in principle as the addition of 12 oranges, 7 oranges, and 18 oranges.

Subtraction: $\frac{3}{7}-\frac{1}{4}=\frac{12}{28}-\frac{7}{28}=\frac{5}{28}$.

Fractions can be compared by converting them into equivalent fractions with a common numerator (L.C.M.).

The least common multiple of the numerators of the fractions $\frac{2}{5}$, $\frac{3}{7}$, and $\frac{5}{14}$ is 30. Now

$$\frac{2}{5}=\frac{2\times 15}{5\times 15}=\frac{30}{75}, \qquad \frac{3}{7}=\frac{3\times 10}{7\times 10}=\frac{30}{70}, \qquad \frac{5}{14}=\frac{5\times 6}{14\times 6}=\frac{30}{84}.$$

Of fractions having a common numerator the greatest is that which has the smallest denominator; for, the smaller the denominator is, the greater are the equal parts into which unity is divided. Hence the ascending order is $\frac{30}{84}$, $\frac{30}{75}$, $\frac{30}{70}$; that is, the ascending order of the original fractions is $\frac{5}{14}$, $\frac{2}{5}$, $\frac{3}{7}$.

It is obvious that this method may be adopted only when comparison of values alone is required, and the same remark applies to the following method:

Since dividing the terms of a fraction by the same number does not alter the value of the fraction, we may divide by its numerator the terms of each of the fractions compared, thus:

$$\frac{2}{5}, \frac{3}{7}, \frac{5}{14} \text{ may be written } \frac{1}{2\frac{1}{2}}, \frac{1}{2\frac{1}{3}}, \frac{1}{2\frac{4}{5}},$$

from which it is seen that the values of the fractions are as determined by the other methods.

It is easy to infer from what has been done that:

(1) *When the numerator of a fraction is increased, the value of the fraction is increased.*

(2) *When the denominator of a fraction is increased, the value of the fraction is decreased.*

The following results should be carefully noted:

$$(1) \qquad \frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}.$$

$$(2) \qquad \frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}.$$

Hence $$\frac{3}{4}+\frac{2}{3}=\frac{3\times 3+2\times 4}{4\times 3}=\frac{9+8}{12}=\frac{17}{12}=1\frac{5}{12};$$

and $$\frac{3}{4}-\frac{2}{3}=\frac{3\times 3-2\times 4}{4\times 3}=\frac{9-8}{12}=\frac{1}{12}.$$

These results are most useful in mental computations.

EXAMPLES. XXV.

(a) Arrange in ascending order of magnitude and find the sum of:

1. $\frac{2}{3}, \frac{5}{6}, \frac{3}{8}$.
2. $\frac{5}{8}, \frac{11}{21}, \frac{15}{32}$.
3. $\frac{4}{5}, \frac{16}{17}, \frac{76}{85}$.
4. $\frac{7}{9}, \frac{17}{18}, \frac{80}{99}$.
5. $\frac{12}{17}, \frac{43}{51}, \frac{129}{136}$.
6. $\frac{11}{13}, \frac{35}{39}, \frac{95}{104}$.
7. $\frac{6}{7}, \frac{3}{8}, \frac{4}{5}, \frac{3}{4}$.
8. $\frac{2}{5}, \frac{7}{15}, \frac{11}{21}, \frac{2}{3}$.

A comparison of fractions is a comparison of ratios. Thus $\frac{2}{3}$ expresses the ratio of 2 to 3, and $\frac{3}{4}$ the ratio of 3 to 4: which is the greater ratio?

$$\frac{2}{3}=\frac{8}{12}, \text{ and } \frac{3}{4}=\frac{9}{12}.$$

The equivalent values show that $\frac{3}{4}$ expresses the greater ratio.

Again, since $\frac{8}{12}$ and $\frac{9}{12}$ represent the same denomination (twelfths), the fractions are compared when their numerators are compared;

$\therefore$ $\frac{8}{12}$ is to $\frac{9}{12}$ as 8 to 9;

that is, $\frac{2}{3}$ is to $\frac{3}{4}$ as 8 to 9,

which means that $\frac{2}{3} = \frac{8}{9}$ of $\frac{3}{4}$,

and that $\frac{3}{4} = \frac{9}{8}$ of $\frac{2}{3}$.

Prove the truth of these results.

EXAMPLES. XXVI.

(*a*) Which is the greater of each of the following pairs of ratios:

1. 1 : 2 or 2 : 3?
2. 5 : 14 or 11 : 28?
3. 11 : 12 or 17 : 18?
4. 7 : 16 or 5 : 12?
5. 7 : 8 or 5 : 6?
6. 17 : 18 or 11 : 12?
7. $\frac{2}{3} : \frac{5}{8}$ or $\frac{2}{3} : \frac{1}{2}$?
8. $7\frac{1}{2} : 10$ or 3 : 4?
9. $\frac{4}{5} : 2\frac{1}{2}$ or $\frac{3}{8} : 1\frac{1}{3}$?
10. $\frac{7}{8} : 3\frac{1}{3}$ or $1\frac{1}{2} : \frac{5}{12}$?

(*b*) Find (1) the sum and (2) the difference of:

1. $\frac{1}{5}$ and $\frac{1}{6}$.
2. $\frac{6}{7}$ and $\frac{5}{9}$.
3. $\frac{7}{10}$ and $\frac{11}{12}$.
4. $\frac{7}{15}$ and $\frac{19}{45}$.
5. $\frac{c}{d}$ and $\frac{e}{f}$.
6. $\frac{x}{y}$ and $\frac{r}{q}$.
7. $\frac{2a}{3b}$ and $\frac{4c}{5d}$.
8. $\frac{5}{9}x$ and $\frac{4}{7}x$.

(*c*) **1.** What must be added to $\frac{2}{3}$ to make $\frac{13}{16}$? to $\frac{2}{5}x$ to make $\frac{13}{15}x$?

2. If $\frac{19}{21}$ of a plank be cut away, what part of it is left?

3. Three men own a ship; *A* owns $\frac{2}{7}$ of it, and *B* $\frac{2}{5}$ of it: what part does *C* own?

4. A man has £x; he spends $\frac{2}{9}$ of it in sheep and $\frac{3}{7}$ of it in lambs: what part of his money now remains?

5. x has the same ratio to 28 that 5 has to 7: find the value of x.

6. $\frac{7}{18}$ of a pole is painted red, $\frac{5}{27}$ of it is painted blue, and the rest is painted white: what part of it is painted white?

7. A man undertakes to plough a field of x acres in 3 days; he ploughs $\frac{3}{8}$ of it the first day and $\frac{7}{15}$ of it the next: how much remains to be done on the third day?

8. On squared paper represent the ratios $\frac{5}{9}$, $\frac{17}{27}$, $\frac{43}{54}$.

To find the meaning of $\frac{3}{4}$ of $\frac{5}{8}$.

The accompanying figure $ABCD$ is seen to contain 32 small equal squares; each square is therefore $\frac{1}{32}$ of the whole figure. The part $AEFD$ is seen to contain 20 of the small squares, that is, $\frac{20}{32}$ or $\frac{5}{8}$ of the whole figure $ABCD$, and this is the part of which we have to find $\frac{3}{4}$. Hence to find $\frac{3}{4}$ of the part denoted by $\frac{5}{8}$ is to find $\frac{3}{4}$ of the part denoted by $\frac{5}{8}$ of some whole quantity, in the present case $\frac{3}{4}$ of $AEFD$. Now, $DFhg$ is seen to be $\frac{3}{4}$ of $AEFD$, which contains 15 of the small squares or $\frac{15}{32}$ of the whole figure $ABCD$.

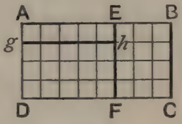


FIG. 8.

The same result is gained symbolically as follows:

$$\frac{3}{4} \text{ of } \frac{5}{8} = \frac{3\times 5}{4\times 8}, \text{ that is, } \frac{\text{product of numerators}}{\text{product of denominators}}$$

$$= \tfrac{15}{32}, \text{ that is, } \tfrac{15}{32} \text{ of the quantity } ABCD.$$

A fraction of a fraction is called a *compound fraction*, which is reduced to a simple fraction as above.

General form: $$\frac{a}{b} \text{ of } \frac{c}{d} = \frac{ac}{bd}.$$

If the numerators and denominators have common factors, the factors should be cancelled out before the multiplication is performed, as:

$$\tfrac{4}{5} \text{ of } \tfrac{15}{16} \text{ of } \tfrac{2}{3} = \tfrac{4\times 15\times 2}{5\times 16\times 3}$$
$$= \tfrac{1}{2},$$

that is, $\frac{1}{2}$ of the quantity of which the last member ($\frac{2}{3}$) of the compound fraction is a part.

The common factors 5, 4, 3, 2 are cancelled out.

General form: $$\frac{a}{b} \text{ of } \frac{c}{d} \text{ of } x = \frac{ac}{bd} \text{ of } x = \frac{x\times ac}{bd}.$$

Thus, if $a=7$, $b=8$, $c=10$, $d=21$, and $x=£24$,

$$\text{then } \frac{a}{b} \text{ of } \frac{c}{d} \text{ of } x = \tfrac{7}{8} \text{ of } \tfrac{10}{21} \text{ of } £24$$
$$= \tfrac{5}{12} \text{ of } £24$$
$$= \frac{£24\times 5}{12}$$
$$= £10.$$

EXAMPLES. XXVII.

(*a*) Reduce to simple fractions or to mixed numbers :

1. $\frac{3}{4}$ of $\frac{8}{9}$ of $\frac{3}{5}$.
2. $\frac{7}{8}$ of $\frac{15}{14}$ of $\frac{3}{10}$.
3. $\frac{5}{6}$ of $\frac{9}{10}$ of $\frac{2}{5}$ of $\frac{1}{2}$.
4. $\frac{5}{9}$ of $\frac{18}{95}$ of $\frac{19}{4}$ of $\frac{5}{2}$.

(*b*) Find the value of :

1. $\frac{3}{7}$ of 18*s*. 1*d*.
2. $\frac{3}{4}$ of $\frac{5}{6}$ of £8. 16*s*. 8*d*.
3. $\frac{2}{5}$ of $\frac{10}{13}$ of £17. 17*s*. 6*d*.
4. $\frac{17}{20}$ of $\frac{15}{34}$ of 8 tons 4 cwt.
5. $\frac{m}{n}$ of $\frac{n}{p}$ of £x.
6. $\frac{a}{b}$ of $\frac{b}{c}$ of $\frac{c}{a}$ of £y.
7. $\frac{p}{q}$ of $\frac{q}{r}$ of $\frac{2}{3}$ of $\frac{r}{p}$ of x lbs.
8. $\frac{a}{c}$ of $\frac{bc}{a}$ of $\frac{r}{b}$ of y oz.

(*c*) If $a=2$, $b=3$, $c=4$, $d=5$, find the values of :

1. $\frac{a}{b}$ of $\frac{c}{d}$ of 15 guineas.
2. $\frac{a}{b}$ of $\frac{b}{c}$ of $\frac{c}{d}$ of 25 cwt.
3. $\frac{a^2}{c^2}$ of $\frac{b^2}{d^2}$ of $\frac{c}{a}$ of 25 cwt.
4. $\frac{ab}{cd}$ of $\frac{c^2d^2}{ab}$ of $\frac{c}{d}$ of $\frac{1}{2}$ a ton.

Mixed Operations.

Ex. 1.

$$\begin{aligned} & 10\tfrac{5}{21}+\tfrac{5}{9} \text{ of } 4\tfrac{1}{7} \\ &=10\tfrac{5}{21}+\tfrac{5}{9} \text{ of } \tfrac{29}{7} \\ &=10\tfrac{5}{21}+\tfrac{145}{63} \\ &=10\tfrac{5}{21}+2\tfrac{19}{63} \\ &=12\tfrac{15+19}{63} \\ &=12\tfrac{34}{63}. \end{aligned}$$

Ex. 2.

$$\begin{aligned} & 10\tfrac{5}{21}-\tfrac{5}{9} \text{ of } 4\tfrac{1}{7} \\ &=10\tfrac{5}{21}-\tfrac{5}{9} \text{ of } \tfrac{29}{7} \\ &=10\tfrac{5}{21}-\tfrac{145}{63} \\ &=10\tfrac{5}{21}-2\tfrac{19}{63} \\ * \quad &=8\tfrac{15-19}{63} \\ &=7\tfrac{63+15-19}{63} \\ &=7\tfrac{59}{63}. \end{aligned}$$

EXAMPLES. XXVIII.

(*a*)

1. $\frac{1}{2}-\frac{1}{3}-\frac{1}{5}+\frac{1}{4}+\frac{7}{10}$.
2. $3\frac{1}{2}+2\frac{1}{12}-7\frac{19}{24}+3\frac{3}{16}$.
3. $3\frac{1}{2}-1\frac{7}{8}+4\frac{5}{12}-2\frac{11}{30}$.
4. $8\frac{1}{2}-4\frac{2}{3}+\frac{3}{5}-1\frac{1}{6}$ of $\frac{2}{5}$.
5. $2\frac{1}{4}$ of $\frac{7}{8}+3\frac{1}{2}-1\frac{5}{7}$ of $\frac{175}{144}$.
6. $2\frac{1}{3}$ of $5\frac{1}{6}-1\frac{7}{8}$ of $6\frac{1}{15}$.
7. $13\frac{1}{2}$ of $\frac{5}{9}+\frac{12}{11}$ of $\frac{22}{25}-7\frac{24}{25}$.
8. $2\frac{2}{5}$ of $\frac{7}{9}+\frac{5}{7}$ of $3\frac{2}{11}$ of $1\frac{2}{15}-\frac{6}{11}$ of $1\frac{4}{9}$ of $2\frac{4}{5}$.

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* Since 19 cannot be taken from 15, we change 1 of the 8 units into sixty-thirds as shown in the next step of the working.

Find the difference between :

9. $\frac{3}{16}$ of $1\frac{3}{28}$ and $\frac{9}{56}$.

10. $\frac{9}{10}$ of $\frac{21}{50}$ and $\frac{9}{10}$ of $\frac{1}{50}$.

11. $2\frac{1}{2}$ of $\frac{3}{5}$ and $\frac{1}{9}$ of $\frac{3}{4}$.

12. $3\frac{1}{6}-2\frac{5}{27}$ and $\frac{2}{3}-\frac{1}{54}$.

(*b*)

1. Add together $1\frac{1}{15}$ of 2*s*. 6*d*., $\frac{7}{16}$ of £1, $\frac{1}{4}$ of £1. 12*s*. 4*d*., and $\frac{1}{30}$ of 15*s*.

2. Add together $\frac{1}{3}$ of 2*s*. 6*d*., $\frac{1}{6}$ of a guinea, $\frac{4}{15}$ of £1, and $\frac{1}{75}$ of £1. 5*s*.

3. Find the value of $1\frac{1}{3}$ of a guinea $+\frac{5}{8}$ of £1 $+\frac{5}{16}$ of half-a-crown $+29\frac{19}{20}$ of $7\frac{1}{2}$*d*.

4. Find the value of $\frac{3}{16}$ of a florin $+\frac{3}{5}$ of 2*s*. 6*d*. $-\frac{1}{10}$ of a crown $-\frac{3}{4}$ of £1 $+\frac{1}{2}$ of 2 guineas.

5. Find the sum of $\frac{4}{5}$ acre $+\frac{1}{36}$ ro., and $\frac{8}{9}$ po.

6. Find the sum of $\frac{4}{9}$ cwt., $\frac{3}{7}$ qr., and $4\frac{3}{8}$ lbs.

7. Find the value of $\frac{3}{8}$ of £1. 10*s*. $+\frac{2}{3}$ of $1\frac{1}{2}$ of $\frac{4}{5}$ of £1. 5*s*. $-\frac{45}{80}$ of £1.

8. Find the value of $\frac{2}{5}$ of £33. 5*s*. 10*d*. $-\frac{4}{11}$ of £31. 2*s*. 5*d*.

9. Find the value of $\frac{3}{14}$ of 3*s*. 6*d*. $+\frac{2}{7}$ of £11. 7*s*. 6*d*. $-\frac{5}{8}$ of £4. 17*s*. 4*d*.

10. Find the sum of $\frac{16}{23}$ of 5*s*. $3\frac{1}{4}$*d*., $\frac{13}{19}$ of 9*s*. $10\frac{3}{4}$*d*., and $\frac{17}{21}$ of 11*s*. $9\frac{3}{4}$*d*.

We express one number as a fraction of another when we make it the numerator of a fraction of which the other is the denominator.

Thus, 3 expressed as a fraction of 4 is $\frac{3}{4}$, which means that 3 is $\frac{3}{4}$ of 4 ; 7 expressed as a fraction of 9 is $\frac{7}{9}$, which means that 7 is $\frac{7}{9}$ of 9 ; and so on.

Obviously, 10 lbs. expressed as a fraction of 1 cwt. is $\frac{10}{112}$ or $\frac{5}{56}$ of a cwt., and 3 inches expressed as a fraction of 1 yard is $\frac{3}{36}$ or $\frac{1}{12}$ of a yard.

The student will see, therefore, that when we express one quantity as a fraction of another, we find the ratio of the one to the other. Hence both quantities must be of *the same kind* and expressed in *the same denomination.*

When one quantity is expressed as a fraction of another, it is said to be expressed in terms of that other. Above, we have expressed 3 in terms of 4, 7 in terms of 9, 10 lbs. in terms of a cwt. and 3 inches in terms of a yard.

a expressed in terms of b is $\dfrac{a}{b}$.

EXAMPLES. XXIX.

1. What fraction of $\frac{1}{4}$ mile is $1\frac{1}{2}$ ft.?
2. Express 2 yds. 3 ins. as a fraction of 1 mile 30 poles.
3. Reduce $\frac{3}{4}$ of a yd. to the fraction of a mile.
4. Express 17 yds. 1 ft. $7\frac{1}{2}$ ins. in terms of half-a-mile.
5. Reduce 4 ozs. 7 dwts. 12 grs. to the fraction of 5 ozs. troy.
6. What is the ratio of a ton to 2 qrs. 27 lbs.?
7. Reduce $5\frac{1}{2}$ gals. to the fraction of $1\frac{3}{8}$ pks.
8. What fraction of 3 sq. poles is 2 sq. poles 22 yds.?
9. Reduce 3 acres 1 ro. 20 poles to the fraction of 21 acres.
10. Express $2\frac{1}{2}$ pints in terms of $3\frac{1}{3}$ gals.
11. Express £1. 9*s*. as the fraction of $4\frac{7}{20}$ of £1.
12. What part of £9 is $\frac{1}{2}$ of $\frac{9}{10}$ of half-a-crown?
13. Reduce 22 days 4 hrs. 35 mins. 42 secs. to the fraction of 34 days 20 hrs. 56 mins. 6 secs.
14. Express x yards in terms of y furlongs.
15. Express x metres in terms of y centimetres.

Miscellaneous Examples.

Ex. 1. *A man who owns $\frac{3}{8}$ of a property leaves $\frac{2}{5}$ of his share to his daughter and the rest to his son: find the son's share of the property.*

He leaves $\frac{2}{5}$ of his share to his daughter;

$\therefore$ he leaves $\frac{3}{5}$ of his share to his son;

$\therefore$ his son's share $=\frac{3}{5}$ of $\frac{3}{8}$ of the property

$=\frac{9}{40}$ of the property.

Solution by another method, but not a good one.

Daughter's share $=\frac{2}{5}$ of $\frac{3}{8}=\frac{3}{20}$ of the property;

$\therefore$ son's share $=\frac{3}{8}-\frac{3}{20}=\frac{15-6}{40}$ of the property

$=\frac{9}{40}$ of the property.

The solution of this example introduces the subject of *complements.* One fraction is said to be the *complement* of another when the sum of the two fractions is unity. If a man parts with $\frac{2}{5}$ of anything, no matter in what terms the thing is expressed, he will have $\frac{3}{5}$ of the thing left; and, if he parts with $\frac{3}{23}$ of a thing, he will have $\frac{20}{23}$ of it left. The sum of $\frac{2}{5}$ and $\frac{3}{5}$ is 1, and so is the sum of $\frac{3}{23}$ and $\frac{20}{23}$. Hence each of the fractions $\frac{2}{5}$ and $\frac{3}{5}$ is the complement of the other, and each of the fractions $\frac{3}{23}$ and $\frac{20}{23}$ is the complement of the other. The properties of complements enable us to dispense with tedious subtractions in examples like the foregoing. The following is a good example of this class of problem.

Ex. 2. *A man owning* $\frac{1}{2}$ *of* $\frac{3}{7}$ *of a property sold* $\frac{2}{5}$ *of his share; he then sold* $\frac{3}{8}$ *of what was then left; he afterwards sold* $\frac{2}{3}$ *of what was then left; and finally he sold* $\frac{9}{10}$ *of what was then left: what part of the property was then left to him?*

He sold $\frac{2}{5}$ of $(\frac{1}{2}$ of $\frac{3}{7})$ of the property;

$\therefore$ he then had $\frac{3}{5}$ of $(\frac{1}{2}$ of $\frac{3}{7})$ of the property left;

of this he sold $\frac{3}{8}$;

$\therefore$ he then had $\frac{5}{8}$ of $(\frac{3}{5}$ of $\frac{1}{2}$ of $\frac{3}{7})$ of the property left;

of this he sold $\frac{2}{3}$;

$\therefore$ he then had $\frac{1}{3}$ of $(\frac{5}{8}$ of $\frac{3}{5}$ of $\frac{1}{2}$ of $\frac{3}{7})$ of the property left;

of this he sold $\frac{9}{10}$;

$\therefore$ he then had $\frac{1}{10}$ of $(\frac{1}{3}$ of $\frac{5}{8}$ of $\frac{3}{5}$ of $\frac{1}{2}$ of $\frac{3}{7})$ of the property left;

which $= \frac{3}{1120}$ of the property.

The work could be shortened by letting x represent the property.

Ex. 3. *I spent* $\frac{1}{3}$ *of my money, then* $\frac{5}{6}$ *of what was left, and finally* $\frac{1}{2}$ *of what then remained; the amount remaining was £5; how much had I at first?*

After spending $\frac{1}{3}$ of my money, I had $\frac{2}{3}$ of it left;

$\therefore$ in the second case I spent $\frac{5}{6}$ of $\frac{2}{3}$ of it,

and then had $\frac{1}{6}$ of $\frac{2}{3}$ of it left;

$\therefore$ in the third case I spent $\frac{1}{2}$ of $(\frac{1}{6}$ of $\frac{2}{3})$ of it,

and then had $\frac{1}{2}$ of $(\frac{1}{6}$ of $\frac{2}{3})$ of it left.

Now, $\frac{1}{2}$ of $(\frac{1}{6}$ of $\frac{2}{3}) = \frac{1}{18}$;

$\therefore$ $\frac{1}{18}$ of my money $=$ £5;

$\therefore$ my money $=$ £5 $\times$ 18 (both sides $\times$ 18)

$=$ £90.

Or thus, using x to represent the money in £, we have

1st remainder $\frac{2}{3}x$,

2nd remainder $\frac{1}{6}$ of $\frac{2}{3}x$,

3rd remainder $\frac{1}{2}$ of $\frac{1}{6}$ of $\frac{2}{3}x = \frac{1}{18}x$;

$\therefore$ $\frac{1}{18}x =$ £5;

$\therefore$ $x =$ £5 $\times$ 18

$=$ £90.

Notice that $18 = \frac{18}{1}$, which is the fraction $\frac{1}{18}$ inverted. An inverted fraction is called the *reciprocal* of the original fraction.

Ex. 4. *Three-sevenths of a number exceeds two-ninths of it by 26: find the number.*

Let x represent the number;

$$\text{then } \tfrac{3}{7}x - \tfrac{2}{9}x = 26;$$

The L.C.M. of the denominators 7 and 9 is 63; hence multiplying both sides of the equation by 63 will clear it of fractions.

$$63\left(\tfrac{3}{7}x - \tfrac{2}{9}x\right) = 26 \times 63;$$
$$\therefore\ 27x - 14x = 26 \times 63;$$
$$\therefore\ 13x = 26 \times 63;$$
$$\therefore\ x = \tfrac{26 \times 63}{13};$$
$$= 126;$$
$$\therefore \text{ the number is } 126.$$

[PROOF. $\tfrac{3}{7}$ of $126 - \tfrac{2}{9}$ of $126 = 54 - 28 = 26$.]

Ex. 5. *A certain fraction becomes $\tfrac{1}{2}$ when 1 is added to its numerator, and $\tfrac{1}{3}$ when 1 is added to its denominator: find the fraction.*

Let $\dfrac{x}{y}$ represent the fraction;

$$\text{then } \frac{x+1}{y} = \frac{1}{2}, \text{ and } \frac{x}{y+1} = \frac{1}{3},$$
$$\text{that is, } 2x + 2 = y, \text{ and } 3x = y + 1,$$
$$\text{that is, } x = \frac{y-2}{2}, \text{ and } x = \frac{y+1}{3};$$
$$\therefore\ \frac{y-2}{2} = \frac{y+1}{3};$$
$$\therefore\ 3y - 6 = 2y + 2;$$
$$\therefore\ y = 8,$$
$$\text{and } 3x = y + 1 = 8 + 1 = 9;$$
$$\therefore\ x = 3;$$
$$\therefore \text{ required fraction is } \frac{3}{8}.$$

$$\left[\text{PROOF. } \frac{3+1}{8} = \frac{1}{2}, \text{ and } \frac{3}{8+1} = \frac{3}{9} = \frac{1}{3}.\right]$$

EXAMPLES. XXX.

[Problems to be worked with and without the aid of x.]

1. A boy loses $\tfrac{3}{8}$ of his marbles in one game and $\tfrac{2}{7}$ of the remainder in another, and then has 50 left: find his total loss.

2. After selling $\tfrac{7}{11}$ of his sheep and then $\tfrac{4}{5}$ of the remainder, a man had 220 left: how many had he at first?

3. A post is $\frac{1}{3}$ of its length in the ground, $\frac{2}{9}$ of its length in the water, and the rest of it, 24 feet, is above water: what is the length of the post?

4. The difference between $\frac{7}{15}$ and $\frac{29}{45}$ of a number is 32: find the number.

5. The sum of two times a number and $\frac{2}{3}$ of the number is 208: what is the number?

6. If $\frac{1}{3}$ of $\frac{2}{11}$ of my money were multiplied by 5 and £25 be taken from the product, I should have £75: how much have I?

7. Find the value of $\frac{4}{5}$ of $\frac{5}{19}$ of £45. 7*s.* 3*d.* + $\frac{2}{3}$ of $\frac{3}{8}$ of a crown.

8. Find the value of $\frac{11}{24}$ of $\frac{56}{77}$ of $10x$ metres $+\,y$ decimetres.

9. Find the value of $\frac{7}{8}$ of $\frac{16}{21}$ of $\frac{2}{3}$ of 27 tons 6 cwt. 3 qrs. 18 lbs.

10. Express 74359 in index notation.

11. Find the value of $2^3+3^2+4^4+10^5\div10^4$.

12. Resolve 1323 into its prime factors. Define *prime factor*.

13. Show on squared paper the meaning of $\frac{3}{4}$ of $\frac{1}{2}$.

14. Which is the greater of the two ratios $\frac{9}{17}$ and $\frac{27}{68}$?

15. To the square of $\frac{5}{9}$ add the square of $\frac{1}{3}$.

16. Why must fractions be brought to equivalent fractions with a common denominator before they can be added or subtracted?

17. Find the value of x in the following equations:

$$(a)\quad \frac{xy}{b}=r+q,$$

$$(b)\quad \frac{6(x-2)}{5}-\frac{8(x-1)}{7}=10.$$

18. How many lbs. of sugar at y pence per lb. can be bought with £x?

19. I cut y feet from a log x yards long: what is the value of the remainder at b shillings per foot?

20. How many square chains are there in x acres?

21. I set out on a journey with £a in my purse: on the first day I spent $\frac{2}{3}$ of it, on the second day $\frac{1}{4}$ of what remained, and on the third day $\frac{3}{8}$ of what then remained: what part of my money had I then left?

22. Find a number such that the sum of its seventh and sixth parts is 13.

23. If 5 be added to the sum of $\frac{1}{3}$ and $\frac{1}{4}$ of a number the sum is 19: find the number.

24. Represent on squared paper the ratios $\frac{2}{3}$, $\frac{7}{12}$, $\frac{5}{8}$, giving to each the denominator 24.

25. Find by factors the square root of 5184, and the cube root of 729.

26. Given that $\frac{1}{2}x+\frac{1}{3}y=8$, and that $\frac{1}{3}x+\frac{1}{2}y=7$, to find the values of x and y.

27. If 4 be taken from each of the terms of a fraction, the fraction becomes $\frac{3}{4}$; and, if 1 be added to the numerator and 4 added to the denominator, it becomes $\frac{2}{3}$: find the fraction.

28. The value of 5 fowls and 7 geese is £3. 4*s.*, and at the same prices the value of 3 fowls and 8 geese is £3. 5*s.*: find the value of each.

Multiplication.

Ex. 1. *To show that* $\frac{4}{5}\times 5=\frac{4\times 5}{5}=4$.

$$\tfrac{4}{5}=4 \text{ fifths};$$

hence $\quad \frac{4}{5}\times 5=5$ times 4 fifths $=20$ fifths $=\frac{20}{5}=4$.

And always, if c is a whole number, then $\dfrac{a}{b}\times c=\dfrac{ac}{b}$.

A fraction is multiplied by a whole number when its numerator is multiplied by the number.

$$\text{Again,}\quad \left.\begin{aligned}\tfrac{2}{3}\times 3&=\tfrac{2\times 3}{3}=2;\\ \text{and}\ \tfrac{2}{3\div 3}&=\tfrac{2}{1}=2;\end{aligned}\right\}\quad \text{similarly}\ \left.\begin{aligned}\tfrac{5}{6}\times 6&=\tfrac{5\times 6}{6}=5;\\ \text{and}\ \tfrac{5}{6\div 6}&=\tfrac{5}{1}=5.\end{aligned}\right\}$$

If c is a whole number, then $\dfrac{a}{b}\times c=\dfrac{a}{b\div c}$.

Dividing the denominator of a fraction by a whole number is equivalent to multiplying the numerator by the number.

Ex. 2. *To show that* $\dfrac{2}{3}\times\dfrac{4}{7}=\dfrac{8}{21}=\dfrac{\text{product of the numerators}}{\text{product of the denominators}}$.

The underlying idea of multiplication is repetition of the multiplicand, the multiplier denoting the number of repetitions.

It is clear, however, that the multiplier $\frac{4}{7}$, being less than 1, denotes, not repetition of $\frac{2}{3}$, but partition of repetition of $\frac{2}{3}$, that is, $\frac{1}{7}$ of 4 times $\frac{2}{3}$, the numerator 4 of $\frac{4}{7}$ denoting repetition and the denominator 7 partition.

Now, $\frac{1}{7}$ of 4 times $\frac{2}{3}=\frac{2\times 4}{3\times 7}=\frac{8}{21}$.

Hence, fractions connected by the sign × are treated in precisely the same manner as those connected by "of."

It follows from the foregoing considerations that, when a quantity is multiplied by a *proper* fraction, the product is less than the original quantity. Show that the contrary is the case when the quantity is multiplied by an *improper* fraction.

Ex. 3. *Find the product of* $\frac{3}{4}$ of $\frac{7}{9}$ and $3\frac{1}{29}$ of $\frac{5}{11}$.

$$\begin{aligned}\tfrac{3}{4} \text{ of } \tfrac{7}{9} \times 3\tfrac{1}{29} \text{ of } \tfrac{5}{11} &= \tfrac{3}{4} \text{ of } \tfrac{7}{9} \times \tfrac{88}{29} \text{ of } \tfrac{5}{11}\\ &= \frac{3 \times 7 \times 88 \times 5}{4 \times 9 \times 29 \times 11}\\ &= \frac{7 \times 2 \times 5}{3 \times 29}\ \text{[factors common to numerator and denominator cancelled]}\\ &= \frac{70}{87}.\end{aligned}$$

Ex. 4. *A rectangular table is 7 feet 4 inches long and 4 feet 7 inches broad: what is the area of the top?*

$$\text{Area} = l \times b.$$

$$7 \text{ ft. } 4 \text{ inches} = 7\tfrac{1}{3} \text{ ft.}, \ 4 \text{ ft. } 7 \text{ inches} = 4\tfrac{7}{12} \text{ ft.};$$

$$\begin{aligned}\therefore\ \text{area} &= (7\tfrac{1}{3} \times 4\tfrac{7}{12}) \text{ square ft.}\\ &= (\tfrac{22}{3} \times \tfrac{55}{12}) \text{ square ft.}\\ &= 33\tfrac{11}{18} \text{ square ft.}\end{aligned}$$

The following should be carefully observed in the working of examples involving mixed operations:

(1) The signs × and ÷ connect factors, and the signs + and − connect separate terms; factors must be reduced to terms before they can be added or subtracted.

(2) Expressions within brackets are generally simplified before the brackets are removed.

EXAMPLES. XXXI.

(*a*) Find the areas of rectangular surfaces having the following dimensions:

1. 2 ft. 3 inches by 9 inches.

2. 9 yards 1 ft. by 5 yards 2 ft.

3. $10\frac{1}{2}$ chains by $5\frac{1}{4}$ chains.

4. 12 ft. 9 inches by 1 ft. 6 inches.

(*b*) Find the values of:

1. $3\frac{5}{9} \times 1\frac{7}{8}$ of $\frac{3}{10} \times \frac{13}{14}$.

2. $\frac{3}{4}$ of $1\frac{5}{11} \times \frac{2}{7}$ of $8\frac{7}{9}$.

3. $3\frac{1}{3} + 7\frac{1}{8} \times 4\frac{1}{3} - 2\frac{1}{9}$.

4. $3\frac{2}{5} - \frac{3}{8} + \frac{1}{4} \times 1\frac{1}{5}$.

5. $5\frac{3}{8} - 2\frac{1}{8} \times 6\frac{3}{17} + 3\frac{1}{5} + 5\frac{11}{20}$.

6. $4\frac{1}{5} - \frac{1}{3} \times \frac{3}{5} + \frac{1}{2} \times \frac{4}{9}$.

7. $4(3\frac{1}{2} + \frac{1}{9})(4\frac{1}{5} - \frac{7}{10})$.

8. $3(4\frac{1}{3} \text{ of } \frac{5}{3})(6\frac{1}{4} \text{ of } \frac{16}{25})$.

9. $7(\frac{16}{21} - \frac{3}{14})(\frac{1}{3} + 1 + \frac{1}{5})$.

10. $8(\frac{1}{4} + \frac{1}{2} - \frac{1}{10})(2\frac{1}{5} - \frac{19}{25})$.

Multiplying a quantity by an *improper* fraction *increases* the quantity in the ratio of the terms of the fraction.

Thus 9 increased in the ratio of 11 to 9 is $9 \times \frac{11}{9}$ or $\frac{11}{9}$ of 9, that is, 11.

The same example expressed concretely: *If* 9 *things cost* £9, *what is the cost of* 11 *things at the same rate?*

The ratio of the things the cost of which is to be found to those the cost of which is given is $\frac{11}{9}$;

$$\begin{aligned}\therefore \text{ the cost of 11 things} &= \tfrac{11}{9} \text{ of the cost of 9 things}\\ &= \tfrac{11}{9} \text{ of £9}\\ &= \text{£11}.\end{aligned}$$

Multiplying a quantity by a *proper* fraction *decreases* the quantity in the ratio of the terms of the fraction.

Thus 11 decreased in the ratio of 9 to 11 is $11 \times \frac{9}{11}$ or $\frac{9}{11}$ of 11, that is, 9.

The same example expressed in concrete terms: *If* 11 *things cost* £11, *what should be paid for* 9 *things at the same rate?*

The ratio of the things the cost of which is to be found to those the cost of which is given is $\frac{9}{11}$;

$$\begin{aligned}\therefore \text{ the cost of 9 things} &= \tfrac{9}{11} \text{ of the cost of 11 things}\\ &= \tfrac{9}{11} \text{ of £11}\\ &= \text{£9}.\end{aligned}$$

Let x represent any quantity, then,

(1) if $\frac{a}{b}$ is an *improper* fraction,

$\frac{ax}{b}$ represents the quantity x *increased* in the ratio of $\frac{a}{b}$;

(2) if $\frac{a}{b}$ is a *proper* fraction,

$\frac{ax}{b}$ represents x *decreased* in the ratio of $\frac{a}{b}$.

How long will it take 12 *men to do a piece of work which* 8 *men of the same working capacity can do in* 27 *days?*

Is the quantity 27 days to be *increased* or *decreased*? Obviously, 12 men can do the work in less time, and therefore the multiplying ratio must be a *proper fraction*, that is, the time in which the larger group of men can do the work is to the time in which the smaller group can do it as 8 to 12. Hence the quantity 27 days must be reduced in the ratio $\frac{8}{12}$.

$\therefore$ 12 men can do the work in $\frac{8}{12}$ of 27 days,

that is, in 18 days.

If this result is correct, then the fraction or ratio $\frac{18}{27}$ is equal to the fraction or ratio $\frac{8}{12}$. Is it?

In each of the following examples will be found two quantities of the same kind and another that is to be increased or decreased in the ratio of these quantities.

EXAMPLES. XXXII.

1. Increase £35 in the ratio of 47 to 5.

2. Decrease 45 cwt. in the ratio of 7 to 15.

3. Increase £6. 10*s.* in the ratio of 25 to 13.

4. Decrease 10 yards 2 feet in the ratio of 11 to 16.

5. A train goes at the rate of 35 miles an hour: if its rate is increased in the ratio of 7 to 5, what is its increased speed?

6. Two children's ages are as 15 to 18: if the younger is 10 years old, what is the age of the elder?

7. My age is 60, and my son's age is to mine as 2 to 5: what is his age?

8. If 25 horses cost £250, how much, at the same rate, should be given for 15 horses?

9. If 25 lbs. of tea can be bought with £1. 17*s.* 6*d.*, how much should be paid for 1 cwt. 63 lbs.?

10. If x acres of land can be bought for £y, how many acres of it are worth £z?

11. Khartoum is 390 metres above sea-level: if the ratio of the metre to the yard is $\frac{39}{36}$, what, in feet, is the height of the town above sea-level?

12. The heights of two boys are as 7 to 5: if the height of the taller boy is 4 feet 8 inches, what is the height of the shorter?

13. In what time will 25 men do a piece of work which 12 men can do in 3 days?

14. If 24 men could have done a piece of work in 30 days, in what time was the work done if the contractor employed 6 extra men?

15. If 8 men would take 75 days to complete a piece of work, how many men would be required to do it in 40 days?

16. How many men in 19 days would do a piece of work which 171 men could do in 12 days?

17. How many acres will 34 men reap in the time that 10 men take to reap 23 acres?

18. If 15 yds. of silk cost £1. 13*s.* 9*d.*, how much will 20 yds. 1 ft. cost at the same rate?

19. If 18 gallons of beer cost £1. 1*s.*, what is the cost of 11 pints?

20. If $4\frac{1}{8}$ cwts. of sugar cost 21 guineas, what is the price of $195\frac{1}{4}$ lbs.?

Division.

Ex. 1. *To find the value of* $2 \div \frac{2}{3}$.

To divide 2 by $\frac{2}{3}$ is to find how many times the part $\frac{2}{3}$ of a unit is contained in 2 such units.

$$\begin{aligned} 2 \text{ units} &= (2 \times 3) \text{ thirds} \\ &= 6 \text{ thirds}, \end{aligned}$$

and $$\frac{6 \text{ thirds}}{2 \text{ thirds}} = 3;$$

$$\therefore\ 2 \div \tfrac{2}{3} = 3.$$

Now, $2 \times \frac{3}{2} = 3$;

∴ *the quotient = dividend × divisor inverted.*

Or thus:

$$\begin{aligned} 2 \div \tfrac{2}{3} &= \tfrac{6}{3} \div \tfrac{2}{3} \\ &= 6 \div 2 \\ &= 3. \end{aligned}$$

Ex. 2. *To find the value of* $\frac{1}{3} \div 2$.

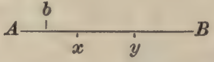
To divide $\frac{1}{3}$ by 2 is to find a half of the part $\frac{1}{3}$ of a unit.

$$\tfrac{1}{2} \text{ of } \tfrac{1}{3} = \tfrac{1}{6};$$

$$\begin{aligned} \therefore\ \tfrac{1}{3} \div 2 &= \tfrac{1}{3} \times \tfrac{1}{2} \\ &= \textit{dividend} \times \textit{divisor inverted}. \end{aligned}$$

Or thus:

$$\begin{aligned} \tfrac{1}{3} \div 2 &= \tfrac{1}{3} \div \tfrac{6}{3} \\ &= 1 \div 6 \\ &= \tfrac{1}{6}. \end{aligned}$$



Let the unit be denoted by the line AB. Divide AB into three equal parts in the points x and y. Then

$$Ax = \tfrac{1}{3} \text{ of } AB.$$

Divide Ax into two equal parts in the point b. Then

$$Ab = \tfrac{1}{2} \text{ of } Ax, \text{ and manifestly also } \tfrac{1}{6} \text{ of } AB;$$

wherefore $$\tfrac{1}{3} \div 2 = \tfrac{1}{3} \times \tfrac{1}{2} = \tfrac{1}{2} \text{ of } \tfrac{1}{3} = \tfrac{1}{6}.$$

In division the quotient may be regarded as expressing either repetition of the divisor (*i.e.* the number of times the divisor is contained in the dividend) or partition of the dividend. Thus $\frac{6}{3}$ may denote either the number of times 6 contains 3 or the third part of 6. Obviously, a fraction divided by a whole number cannot denote repetition in the usual sense, for a part of a unit cannot contain the whole unit; much less can it contain more than 1 unit, for example, 2 as in the present case. Nevertheless, by an extension of meaning the notion "times" is in mathematics made to include "parts of times." Thus 9 contains 2 four times with enough left over to make $\frac{1}{2}$ of another repetition. It is therefore said to contain 2 four and a half times. In a similar way $\frac{1}{3}$ may be regarded as containing 2 one-sixth of a time.

It should be remembered too that the division of one quantity by another also denotes the ratio of the first to the second. Thus $2 \div \frac{2}{3}$ expresses the ratio of 2 to $\frac{2}{3}$, namely, as we have seen, $\frac{3}{1}$ or 3 to 1, sometimes written $3:1$; and $\frac{1}{3} \div 2$ expresses the ratio of $\frac{1}{3}$ to 2, namely, $\frac{1}{6}$ or 1 to 6.

Ex. 3. *To find the value of* $\frac{2}{3} \div \frac{5}{7}$.

$$\frac{2}{3} \div 5 = \frac{1}{5} \text{ of } \frac{2}{3} \text{ (Ex. 2)},$$

$$\text{but 5 is 7 times } \frac{5}{7};$$

$$\therefore \text{ the quotient } \frac{1}{5} \text{ of } \frac{2}{3} \text{ is only } \frac{1}{7} \text{ of the quotient of } \frac{2}{3} \div \frac{5}{7};$$

$$\begin{aligned} \therefore\ \frac{2}{3} \div \frac{5}{7} &= 7 \text{ times } \frac{1}{5} \text{ of } \frac{2}{3} \\ &= \frac{2}{3} \times \frac{7}{5} = \textit{dividend} \times \textit{divisor inverted} \\ &= \frac{14}{15}. \end{aligned}$$

$$\begin{aligned} \text{Again,}\quad \frac{2}{3} \div \frac{5}{7} &= \frac{14}{21} \div \frac{15}{21} \\ &= 14 \div 15 \\ &= \frac{14}{15}, \end{aligned}$$

which is the same result as that obtained by *multiplying the dividend by the divisor inverted.*

Give of the answer as many interpretations as you can.

The student should prove for himself that dividing the numerator of a fraction by a whole number is equivalent to multiplying the denominator by the number.

Ex. 4. *Find the value of* $2\frac{1}{2} - \frac{1}{3} \times 2\frac{1}{2}$ of $\frac{2}{3} \div \frac{3}{2} \times \frac{1}{2} + 3\frac{1}{3}$.

$$\begin{aligned} 2\tfrac{1}{2} - \tfrac{1}{3} \times 2\tfrac{1}{2} \text{ of } \tfrac{2}{3} \div \tfrac{3}{2} \times \tfrac{1}{2} + 3\tfrac{1}{3} &= 2\tfrac{1}{2} - \tfrac{1}{3} \times \tfrac{5}{2} \times \tfrac{2}{3} \times \tfrac{2}{3} \times \tfrac{1}{2} + 3\tfrac{1}{3} \\ &= 2\tfrac{1}{2} - \tfrac{5}{27} + 3\tfrac{1}{3} \\ &= 5 + \tfrac{1}{2} - \tfrac{5}{27} + \tfrac{1}{3} \\ &= 5\tfrac{27-10+18}{54} \\ &= 5\tfrac{35}{54}. \end{aligned}$$

This example is intended mainly as a study in the meanings and effects of the signs. The student should notice carefully how the second term of the second line is obtained, and that the multiplication and division signs are dealt with in the order in which they occur and before the operations of addition and subtraction are performed.

Ex. 5. *Simplify* $\dfrac{4\frac{1}{2}}{3\frac{1}{3}}$.

The form $\dfrac{4\frac{1}{2}}{3\frac{1}{3}}$ is only another form for $4\frac{1}{2} \div 3\frac{1}{3}$, or for expressing the ratio of $4\frac{1}{2}$ to $3\frac{1}{3}$, and to simplify it is to express its value in the form of a simple fraction, that is, a fraction having both of its terms whole numbers.

$$\frac{4\frac{1}{2}}{3\frac{1}{3}} = \frac{27}{20} = 1\tfrac{7}{20}.$$

(Numerator and denominator multiplied by 6, the L.C.M. of 2 and 3.)

Or thus:

$$\frac{4\frac{1}{2}}{3\frac{1}{3}} = \frac{\frac{9}{2}}{\frac{10}{3}}$$
$$= \frac{9 \times 3}{2 \times 10}$$
$$= \frac{27}{20} = 1\tfrac{7}{20}.$$

Or thus:

$$\frac{4\frac{1}{2}}{3\frac{1}{3}} = 4\tfrac{1}{2} \div 3\tfrac{1}{3}$$
$$= 4\tfrac{1}{2} \times \frac{1}{3\frac{1}{3}}$$
$$= \frac{9}{2} \times \frac{3}{10}$$
$$= \frac{27}{20} = 1\tfrac{7}{20}.$$

The first method is generally the best for purposes of mere simplification, but it is sometimes convenient to write the work in the second or the third form.

Ex. 6. *Simplify* $\dfrac{3\frac{1}{4}}{2\frac{1}{6}} \times \dfrac{2\frac{1}{2} - \frac{1}{6}}{\frac{2}{3} \times \frac{1}{2}}$.

$$\frac{3\frac{1}{4}}{2\frac{1}{6}} \times \frac{2\frac{1}{2} - \frac{1}{6}}{\frac{2}{3} \times \frac{1}{2}} = \frac{78}{52} \times \frac{60 - 4}{8}$$
$$= \frac{3}{2} \times \frac{56}{8}$$
$$= 10\tfrac{1}{2}.$$

The first step in the simplification is the result of multiplying the numerators and denominators by their L.C.M. 24.

The last two examples are called *complex fractions*, that is, fractions divided by fractions.

We have already seen that to divide by a fraction we invert the divisor and multiply. Now a fraction *inverted* is the *reciprocal* of the fraction. Hence we may state the rule for division by a fraction as follows:

Multiply the dividend by the reciprocal of the divisor.

The following results are important in this connexion:

If $\frac{3}{4}x = y$,

then $x = \frac{4}{3}$ of y (both sides $\div \frac{3}{4}$, that is, $\times$ the reciprocal of $\frac{3}{4}$):

and, if $x = \frac{4}{3}$ of y,

then $y = \frac{3}{4}$ of x.

Prove these results on squared paper.

Ex. 7. *If* $\frac{7}{8}$ *of my property is worth* £686, *what is the property worth?*

Value of $\frac{7}{8}$ of the property $=$ £686;

$\therefore$ value of the property $=$ £686 $\times \frac{8}{7}$ (both sides $\div \frac{7}{8}$)

$=$ £784.

Or thus:

Let x represent the value of the property in £,

then $\frac{7}{8}x =$ £686;

$\therefore x =$ £686 $\times \frac{8}{7}$

$=$ £784.

Ex. 8. *The floor of a rectangular room has an area of* 111 *square feet, and its length is* $12\frac{1}{3}$ *feet: find its width.*

$$l \times b = \text{area};$$

$$\therefore b = \frac{\text{area}}{l}$$

$$= \frac{111}{12\frac{1}{3}} \text{ ft.}$$

$$= 9 \text{ ft.}$$

Ex. 9. *What sum is the same fraction of* 4 *shillings that* 9*s.* 1*d. is of* £1. 7*s.* 3*d.?*

Let x represent the sum in pence,

then $\dfrac{x}{4 \times 12} = \dfrac{109}{327} = \dfrac{1}{3}$;

$\therefore x = \dfrac{4 \times 12}{3} d. = 16d. = 1s.\ 4d.$

[PROOF. $\dfrac{1s.\ 4d.}{4s.} = \dfrac{16}{48} = \dfrac{1}{3}$.]

9*s.* 1*d.* = 109*d.*

£1. 7*s.* 3*d.* = 327*d.*

EXAMPLES. XXXIII.

Simplify:

1. $\dfrac{3\frac{1}{2}}{2\frac{1}{7}}$.
2. $\dfrac{5\frac{1}{8}}{1\frac{7}{41}}$.
3. $\dfrac{4\frac{2}{9}}{1\frac{1}{18}}$.
4. $\dfrac{4\frac{1}{3}}{3\frac{3}{26}}$.
5. $\dfrac{14\frac{4}{9}}{11\frac{2}{3}}$.
6. $\dfrac{20\frac{1}{4}}{2\frac{2}{9}}$.
7. $\dfrac{2\frac{4}{7}}{\frac{27}{14}}$.
8. $\dfrac{11\frac{2}{3}}{1\frac{3}{4}}$.

Simplify:

9. $\dfrac{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{2\frac{1}{2}+3\frac{1}{2}+4\frac{1}{2}}$.

10. $\dfrac{\frac{2}{9}\text{ of }1\frac{2}{3}\text{ of }2\frac{2}{5}}{\frac{7}{9}+\frac{5}{12}-\frac{3}{4}}$.

11. $(\frac{6}{7}\text{ of }4)\div(\frac{5}{9}+\frac{2}{3})$.

12. $\frac{6}{7}\text{ of }4\div\frac{5}{9}+\frac{2}{3}$.

13. $\frac{5}{9}\div\frac{6}{7}\text{ of }4+\frac{2}{3}$.

14. $\dfrac{\frac{1}{2}+1\frac{1}{5}+2\frac{1}{6}}{\frac{1}{2}\text{ of }1\frac{3}{5}-\frac{1}{3}\text{ of }\frac{4}{5}}$.

15. $\frac{1}{2}+1\frac{1}{5}+2\frac{1}{6}\div\frac{1}{2}\text{ of }1\frac{3}{5}-\frac{1}{3}\text{ of }\frac{4}{5}$.

16. $\dfrac{5\frac{1}{10}\text{ of }8\frac{1}{6}}{3\frac{1}{11}\text{ of }4\frac{9}{10}}$.

17. $\dfrac{\frac{7}{11}\times3\frac{1}{2}}{\frac{7}{9}-\frac{3}{7}}\div6\frac{48}{55}$.

18. By what number must $8\frac{1}{2}$ be multiplied to produce $15\frac{1}{3}$?

19. By what number must $3\frac{1}{2}$ of $1\frac{1}{3}$ be multiplied to produce $21\frac{1}{2}$?

20. What number divided by $4\frac{1}{3}$ will yield $8\frac{2}{5}$ for quotient?

21. What number divided by $7\frac{1}{2}$ of $4\frac{1}{4}$ will yield $18\frac{3}{8}$ for quotient?

22. A man lost $\frac{5}{6}$ of his capital, and then had £150 left: how much had he to begin with?

23. A coal merchant sold to one customer $\frac{1}{8}$ of his stock, to a second $\frac{1}{3}$ of what remained, and to a third $\frac{1}{10}$ of what then remained; he then had 41 tons 6 cwt. 98 lbs. left: how much coal had he to begin with?

24. *A*'s money is equal to $\frac{2}{3}$ of *B*'s, and between them they have £25: find how much each has.

25. *A* received $\frac{3}{19}$ and *B* $\frac{4}{17}$ of a certain sum of money, and *C* the remainder: if *A* received £107. 14*s*. 9*d*., what did each of the others receive?

26. If x is equal to $\frac{4}{5}$ of y, what is the value of y expressed in terms of x?

27. A roll of paper is $19\frac{1}{2}$ feet long and contains $48\frac{3}{4}$ square feet: find its width.

28. The area of a rectangular plank is $12\frac{1}{4}$ square feet, and its width is 9 inches: what is its length?

29. What sum is the same fraction of £1. 15*s*. that £2. 6*s*. $8\frac{1}{2}$*d*. is of £8. 3*s*. $5\frac{3}{4}$*d*.?

30. What fraction of $\frac{2}{29}$ of £3. 19*s*. 9*d*. is equal to $\frac{2}{5}$ of £1. 2*s*.?

31. $\frac{4}{9}$ of £4. 1*s*. $11\frac{1}{4}$*d*. equals $\frac{2}{15}$ of a certain sum of money: find the sum.

32. $\frac{5}{11}$ of 77 tons 9 cwt. 2 qrs. 14 lbs. equals $\frac{2}{3}$ of my share of a cargo of sugar: find my share.

CHAPTER V.

AVERAGES.

If a number of quantities of *the same kind* be added, and their sum be divided by their number, the quotient gives what is called the *average* or *mean* of the quantities.

Ex. 1. *Find the average of* $7\frac{1}{2}$, $3\frac{1}{4}$, $2\frac{3}{8}$.

$$\begin{aligned}\text{The average} &= \tfrac{1}{3}(7\tfrac{1}{2}+3\tfrac{1}{4}+2\tfrac{3}{8})\\ &= \tfrac{1}{3}(12+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{3}{8})\\ &= \tfrac{1}{3}(12+\tfrac{4+2+3}{8})\\ &= \tfrac{1}{3}\text{ of }13\tfrac{1}{8}\\ &= 4\tfrac{3}{8}.\end{aligned}$$

Generally, (*a*) if there be n quantities of the same kind, $a, b, c, d \ldots$, the average of the quantities is

$$\frac{1}{n}(a+b+c+d\ldots);$$

(*b*) if x be the average of n quantities, the sum of the quantities is nx.

In Ex. 1, for instance, the average is $4\frac{3}{8}$, and n is 3;

$$\therefore \text{ the sum} = 4\tfrac{3}{8}\times 3 = 13\tfrac{1}{8}.$$

To find the average of the whole when the averages of two or more sets are given.

If the average of n quantities is x, the average of m quantities of the same kind y, and the average of p quantities of the same kind z,

$$\text{then the average of the whole } = \frac{nx+my+pz}{n+m+p}.$$

Ex. 2. *I buy 8 lbs. of tea at an average price of 1s. 4d. per lb., 6 lbs. at an average price of 1s. 2d. per lb., and 10 lbs. at an average price of 1s. per lb. : find the average price of the whole.*

$$\text{The average} = \frac{1s.\ 4d. \times 8 + 1s.\ 2d. \times 6 + 1s. \times 10}{8+6+10}$$

$$= \frac{27s.\ 8d.}{24} = 1s.\ 1\tfrac{5}{6}d.$$

Ex. 3. *The average height of three persons is 5 feet $5\frac{1}{3}$ inches, and the height of one of them is 5 feet 8 inches: find the average height of the two others.*

$$\text{Sum of height} = 5 \text{ ft. } 5\tfrac{1}{3} \text{ in.} \times 3 = 16 \text{ ft. } 4 \text{ inches};$$

$$\therefore \text{ average sought} = \frac{16 \text{ ft. } 4 \text{ inches} - 5 \text{ ft. } 8 \text{ inches}}{2}$$

$$= 5 \text{ ft. } 4 \text{ inches.}$$

EXAMPLES. XXXIV.

1. Find the average of the following sets of numbers :

(*a*) $3\frac{1}{8}$, $7\frac{2}{3}$, and $8\frac{5}{12}$; (*b*) $5\frac{4}{9}$, $8\frac{3}{5}$, and $9\frac{19}{45}$.

(*c*) $2\frac{1}{2}$, $4\frac{2}{3}$, $3\frac{1}{4}$, $1\frac{9}{10}$. (*d*) $21\frac{4}{7}$, $35\frac{1}{11}$, $22\frac{1}{7}$, $4\frac{1}{2}$.

2. The average of 7 numbers is $17\frac{5}{6}$: find the sum of the numbers.

3. The average attendance of a school for a week of 5 days was $27\frac{7}{10}$: find the sum of the attendances.

4. 17 articles are bought at 2*s.* 11*d.* each and 18 more of the same kind at 1*s.* 5½*d.* each : what is the average price per article ?

5. The average of 8 numbers is $6\frac{47}{80}$; the average of the three first numbers is $3\frac{11}{15}$, and the average of the two last numbers is 7: find the average of the remaining numbers.

CHAPTER VI.

TIME AND WORK.

Ex. 1. *A can do a piece of work in* 6 *days, B can do it in* 8 *days, and C in* 12 *days: how long will they take to complete it, all working together?*

A can do $\frac{1}{6}$ of the work daily,

B can do $\frac{1}{8}$ of the work daily,

C can do $\frac{1}{12}$ of the work daily;

working together A, B, and C can do $(\frac{1}{6}+\frac{1}{8}+\frac{1}{12})$ of it daily,

that is, $\frac{3}{8}$ of it daily.

The whole work to be done may be denoted by unity or 1; hence as often as $\frac{3}{8}$ is contained in 1, so many days will it take the men working together to do the work. Now, to divide a number by a fraction is to multiply the number by the reciprocal of the fraction, and, obviously, if the number divided by the fraction be 1, the quotient is the reciprocal of the fraction; hence the reciprocal of the fraction denoting the daily work of a number of workers denotes the number of days it will take the workers to do the work;

$\therefore$ A, B, and C can do the work in $\frac{8}{3}$ days,

that is, $2\frac{2}{3}$ days.

And, generally, if $\frac{a}{b}$ denotes the fraction of the work x men can do in any unit of time, then $\frac{b}{a}$ denotes the number of such units of time it will take x men to do the whole work.

Ex. 2. *If A and B together can perform a piece of work in* 10 *days, and A himself can do it in* 18 *days, what time will it take B to do it alone?*

A and B working together can do the work in 10 days;

$\therefore$ they can do $\frac{1}{10}$ of it daily.

A working alone can do $\frac{1}{18}$ of it daily;

$\therefore$ B working alone can do $(\frac{1}{10}-\frac{1}{18})$ of it daily,

that is, $\frac{2}{45}$ of it daily;

$\therefore$ B working alone can do the work in $\frac{45}{2}$ days, or $22\frac{1}{2}$ days.

Ex. 3. *A and B can build a boat in 18 days, and with the assistance of C they can build it in 11 days: in what time can C build it by himself?*

A and B working together can do $\frac{1}{18}$ of the work daily,

and A, B, and C working together can do $\frac{1}{11}$ of it daily;

$\therefore$ C can do $(\frac{1}{11}-\frac{1}{18})$ of it daily, that is, $\frac{7}{198}$ of it daily;

$\therefore$ C can build the boat in $\frac{198}{7}$ days, or $28\frac{2}{7}$ days.

Ex. 4. *A does $\frac{7}{10}$ of a piece of work in 14 days: he then calls in B, and they finish the work in 2 days: how long would B take to do the whole work by himself?*

$\frac{3}{10}$ of the work remains to be done.

A does $\frac{7}{10}$ of the work in 14 days;

$\therefore$ A does $\frac{1}{14}$ of $\frac{7}{10}$ of the work, that is, $\frac{1}{20}$ of it daily,

and therefore $\frac{2}{20}$ or $\frac{1}{10}$ of it in 2 days;

$\therefore$ B's share for the 2 days $=(\frac{3}{10}-\frac{1}{10})$ of the work $=\frac{2}{10}$ of the work;

$\therefore$ B's daily share $=\frac{1}{10}$ of the work;

$\therefore$ B would take 10 days to do the work.

Ex. 5. *A and B can do a piece of work in 10 days, B and C in 15 days, and A and C in 25 days: they all work at it together for 4 days; A then leaves, and B and C go on together for 5 days, and then B leaves: in how many days will C complete the work?*

A's daily work $+$ B's $=\frac{1}{10}$ of the whole,

A's daily work $+$ C's $=\frac{1}{25}$ of the whole;

$\therefore$ 2 times A's $+$ B's $+$ C's $=(\frac{1}{10}+\frac{1}{25})$ of the whole;

but B's $+$ C's $=\frac{1}{15}$ of the whole;

$\therefore$ 2 times A's $=(\frac{1}{10}+\frac{1}{25}-\frac{1}{15})$ of the whole

$=\frac{11}{150}$ of the whole;

$\therefore$ 2 times A's daily work $=\frac{11}{150}$ of the whole;

$\therefore$ A's daily work $=\frac{11}{300}$ of the whole.

Again, A's daily work $+$ B's $=\frac{1}{10}$ of the whole;

$\therefore$ B's daily work $=(\frac{1}{10}-\frac{11}{300})$ of the whole

$=\frac{19}{300}$ of the whole;

also, since B's daily work $+$ C's $=\frac{1}{15}$ of the whole,

$\therefore$ C's daily work $=\left(\frac{1}{15}-\frac{19}{300}\right)$ of the whole

$=\frac{1}{300}$ of the whole.

Now, since A's daily work $=\frac{11}{300}$, B's $=\frac{19}{300}$, and C's $=\frac{1}{300}$ of the whole,

A's $+$ B's $+$ C's $=\frac{11}{300}+\frac{19}{300}+\frac{1}{300}$, or $\frac{31}{300}$ of the whole;

$\therefore$ in 4 days they can do $\frac{31\times 4}{300}$, or $\frac{124}{300}$ of the work;

$\therefore$ $\frac{176}{300}$ of the work remains to be done when A leaves.

B and C now continue the work together for 5 days;

working together they can do $\frac{20}{300}$ of the whole work daily;

$\therefore$ in 5 days they do $\frac{20\times 5}{300}$ of the whole work, that is, $\frac{100}{300}$ of it;

$\therefore$ $\left(\frac{176}{300}-\frac{100}{300}\right)$ of the whole work remains to be done when B leaves;

$\therefore$ C has to do $\frac{76}{300}$ of it, working alone; and, since he can do $\frac{1}{300}$ of it daily, he will finish the work in 76 days.

EXAMPLES. XXXV.

1. A alone can do a piece of work in 11 days, and B alone can do it in 17 days: find how long they would take to do it together.

2. A can dig a garden in 5 days, B in 6 days, and C in 7 days: in what time will the three working together finish it?

3. In the last sum find the time in which A and B together, A and C together, and B and C together could respectively finish the work.

4. A can finish a piece of work in $2\frac{1}{2}$ days, and B in $3\frac{1}{3}$ days: if they work together what part of the work will they finish in $1\frac{3}{7}$ days?

5. If a cistern can be filled by one pipe in 3 hours, by a second in 5 hours, and by a third in 6 hours, in what time will it be filled by all running together?

6. A can reap $\frac{4}{9}$ of a field in $2\frac{2}{3}$ days, and B can reap $\frac{3}{5}$ of it in $4\frac{1}{2}$ days: in what time could A and B working together reap the field?

7. A can do a piece of work in 10 days, and B can do it in 8 days: after A has worked three days, in how many days will A and B finish the work together?

8. A and B together can do a piece of work in 30 days, B by himself can perform the same in 70 days: in what time could A do it alone?

9. A and B can do a piece of work in 12 days: when C joins them they can do it in 9 days: in what time can C do it working alone?

10. A man alone can do in 10 days a piece of work which, if his son helps him, he can do in 6 days: in what time could his son working alone do the work?

11. *A* and *B* can do a piece of work in 5 days, *B* can do it in 6 days, and *C* in 7 days: how much of it can they do jointly in 2 days?

12. *A* digs $\frac{5}{8}$ of a field in 11 days, and then with *B*'s help does the rest in 4 days: how long would it take each working alone to dig the whole field?

13. Three pipes, *A*, *B*, and *C*, connected with a cistern, are opened at 1, 2, and 3 o'clock respectively: *A* alone would fill it in 3 hours, and *B* alone in 4 hours, while *C* can empty it in one hour: at what o'clock will the cistern be empty?

14. A cistern is filled by two spouts in 20 minutes and 24 minutes respectively, and emptied by a tap in 30 minutes: the cistern being empty, what portion of it will be filled in 15 minutes when all three taps are opened together?

15. *A* and *B* can do a piece of work in 12 days, *A* and *C* can do the same work in 15 days, and *B* and *C* in 18 days: in what time could each do it separately?

16. *A* and *B* can do a piece of work in 4 days, *B* and *C* in 6 days, and *A* and *C* in 8 days: in what time could they do it, all working together?

17. *A* and *B* can do a piece of work in 6 days, *B* and *C* can do it in 7 days, and *A*, *B*, and *C* can do it in 4 days: how long will *A* and *C* take to do it?

18. *A* and *B* together do a piece of work in 11 days, *A* and *C* together can do it in 5 days, and *B* and *C* together in 8 days: how long will it take each working alone to do it?

19. If *A* can do as much work in 5 hours as *B* can do in 6 hours, or as *C* can do in 9 hours, how long will it take *A* to complete a piece of work, one-half of which has been done by *B* working 12 hours, and *C* working 24 hours?

20. *A* can do as much work in 2 days as *B* can do in 3 days, and *B* as much in 4 days as *C* in 5 days: in what time could *A*, *B*, and *C* together do a piece of work which *A* can do in 11 days?

CHAPTER VII.

UNEQUAL PARTITION OF QUANTITIES.

Ex. 1. *Divide £11875 among A, B, and C so that as often as A gets £4 B shall get £3, and as often as B gets £6 C shall get £5.*

$$A\text{'s share} = \tfrac{4}{3} \text{ of } B\text{'s};$$

$$\therefore \text{ when } B \text{ gets £6, } A \text{ gets } \tfrac{4}{3} \text{ of £6 or £8};$$

$$\therefore \text{ as often as } A \text{ gets £8, } B \text{ gets £6 and } C \text{ £5};$$

∴ of every sum of £8, £6, and £5 contained in the amount to be divided among A, B, and C, A will get $\frac{8}{19}$, B $\frac{6}{19}$, and C $\frac{5}{19}$; the amount to be divided is £11875;

$$\therefore\ A\text{'s share} = \tfrac{8}{19} \text{ of £11875} = \text{£5000},$$

$$B\text{'s share} = \tfrac{6}{19} \text{ of £11875} = \text{£3750},$$

$$\text{and}\quad C\text{'s share} = \tfrac{5}{19} \text{ of £11875} = \text{£3125}.$$

Ex. 2. *Divide £38. 8s. among A, B, and C, so that B may have two-thirds as much again as A, and C three-fifths of A's share and B's together.*

$$B\text{'s share} = A\text{'s} + \tfrac{2}{3}\ A\text{'s} = 1\tfrac{2}{3}\ A\text{'s}.$$

$$C\text{'s share} = \tfrac{3}{5} \text{ of } \{A\text{'s} + B\text{'s}\};$$

$$\therefore\ C\text{'s share} = \tfrac{3}{5} \text{ of } 2\tfrac{2}{3} \text{ of } A\text{'s} = 1\tfrac{3}{5}\ A\text{'s};$$

$$\therefore\ A\text{'s share} + B\text{'s} + C\text{'s} = (1 + 1\tfrac{2}{3} + 1\tfrac{3}{5}) \text{ of } A\text{'s} = 4\tfrac{4}{15}\ A\text{'s};$$

but their united shares $= £38\frac{2}{5}$; ∴ $4\frac{4}{15}$ A's share $= £38\frac{2}{5}$;

$$\therefore\ A\text{'s share} = \text{£}\frac{38\frac{2}{5}}{4\frac{4}{15}} = \text{£9},$$

$$\text{and, since}\quad B\text{'s share} = 1\tfrac{2}{3}\ A\text{'s},\quad \therefore\ B\text{'s share} = \text{£9} \times 1\tfrac{2}{3} = \text{£15};$$

$$\text{and, since}\quad C\text{'s share} = 1\tfrac{3}{5}\ A\text{'s},\quad \therefore\ C\text{'s share} = \text{£9} \times 1\tfrac{3}{5} = \text{£14. 8s.}$$

Or work thus, using x to represent A's share in £,

$$\text{then}\quad B\text{'s share} = 1\tfrac{2}{3}x = \tfrac{5}{3}x,$$
$$\text{and}\quad C\text{' share} = \tfrac{3}{5}(x + \tfrac{5}{3}x) = \tfrac{3}{5}x + x;$$
$$\therefore\ x + \tfrac{5}{3}x + \tfrac{3}{5}x + x = £38.\ 8s.;$$
$$\therefore\ 15x + 25x + 9x + 15x = £576;$$
$$\therefore\ 64x = £576;$$
$$\therefore\ x = £9,$$

from which the other shares can be easily found.

Ex. 3. *A guinea is divided among A, B, and C: A's share is* $\tfrac{3}{5}$ *of B's; it is also* $\tfrac{2}{5}$ *of B's and C's together; find each one's share of the guinea.*

$$A\text{'s share} = \tfrac{3}{5}\ B\text{'s},\quad A\text{'s share} = \tfrac{2}{5}\text{ of }(B\text{'s} + C\text{'s});$$
$$\therefore\ \tfrac{3}{5}\ B\text{'s} = \tfrac{2}{5}\text{ of }(B\text{'s} + C\text{'s}) = \tfrac{2}{5}\ B\text{'s} + \tfrac{2}{5}\ C\text{'s};$$
$$\therefore\ \tfrac{3}{5}\ B\text{'s} - \tfrac{2}{5}\ B\text{'s} = \tfrac{2}{5}\ C\text{'s};\quad \therefore\ \tfrac{1}{5}\ B\text{'s} = \tfrac{2}{5}\ C\text{'s};$$
$$\therefore\ B\text{'s} = 2\ C\text{'s};\quad \therefore\quad C\text{'s} = \tfrac{1}{2}\ B\text{'s};$$
$$\therefore\ A\text{'s share} + B\text{'s} + C\text{'s} = (\tfrac{3}{5} + 1 + \tfrac{1}{2})\text{ of }B\text{'s} = 2\tfrac{1}{10}\ B\text{'s};$$

but their united shares = 21 shillings;

$$\therefore\ 2\tfrac{1}{10}\ B\text{'s share} = 21\text{ shillings};$$
$$\therefore\ B\text{'s share} = \frac{21}{2\frac{1}{10}}\text{ shillings} = 10\text{ shillings};$$
$$\therefore\ A\text{'s share} = \tfrac{3}{5}\text{ of }10\text{ shillings} = 6\text{ shillings};$$
$$\therefore\ C\text{'s share} = \tfrac{1}{2}\text{ of }10\text{ shillings} = 5\text{ shillings}.$$

Or work thus, using x for B's share in shillings.

$$x = B\text{'s share, then }\tfrac{3}{5}x = A\text{'s share};$$
$$\tfrac{2}{5}\text{ of }(B\text{'s} + C\text{'s}) = A\text{'s share} = \tfrac{3}{5}x;$$
$$\therefore\ \tfrac{2}{5}x + \tfrac{2}{5}\text{ of }C\text{'s share} = \tfrac{3}{5}x;$$
$$\therefore\ \tfrac{2}{5}\text{ of }C\text{'s share} = \tfrac{3}{5}x - \tfrac{2}{5}x = \tfrac{1}{5}x;$$
$$\therefore\ C\text{'s share} = \tfrac{1}{5}x \times \tfrac{5}{2} = \tfrac{1}{2}x;$$
$$\therefore\ x + \tfrac{3}{5}x + \tfrac{1}{2}x = 21\text{ shillings};$$
$$\therefore\ 10x + 6x + 5x = (21 \times 10)\text{ shillings};$$
$$\therefore\ 21x = (21 \times 10)\text{ shillings};$$
$$\therefore\ x = \tfrac{21 \times 10}{21}\text{ shillings} = 10\text{ shillings; and so on.}$$

Ex. 4. *Divide 16s. $8\tfrac{1}{4}$d. among A, B, C, and D, so that A's share may be* $\tfrac{3}{10}$ *of D's, C's share* $\tfrac{3}{10}$ *of A's, and B's share the sum of A's and C's.*

$$C\text{'s share} = \tfrac{3}{10}\ A\text{'s},\quad B\text{'s share} = A\text{'s} + C\text{'s};$$
$$\therefore\ B\text{'s share} = A\text{'s} + \tfrac{3}{10}\ A\text{'s} = 1\tfrac{3}{10}\ A\text{'s};$$
$$\tfrac{3}{10}\ D\text{'s share} = A\text{'s};\quad \therefore\ D\text{'s share} = \tfrac{10}{3}\ A\text{'s} = 3\tfrac{1}{3}\ A\text{'s};$$

$\therefore$ A's share + B's + C's + D's $= \left(1 + 1\frac{3}{10} + \frac{3}{10} + 3\frac{1}{3}\right)$ times A's
$= 5\frac{14}{15}$ times A's;

but their united shares = 16*s.* $8\frac{1}{4}$*d.*;

$\therefore$ $5\frac{14}{15}$ times A's share = 16*s.* $8\frac{1}{4}$*d.*;

$$\therefore\ A\text{'s share} = \frac{16s.\ 8\frac{1}{4}d.}{5\frac{14}{15}} = 2s.\ 9\tfrac{3}{4}d.;$$

$$\therefore\ B\text{'s share} = 2s.\ 9\tfrac{3}{4}d. \times 1\tfrac{3}{10} = 3s.\ 7\tfrac{7}{8}d.;$$

$$\therefore\ C\text{'s share} = 2s.\ 9\tfrac{3}{4}d. \times \tfrac{3}{10} = 10\tfrac{1}{8}d.;$$

$$\therefore\ D\text{'s share} = 2s.\ 9\tfrac{3}{4}d. \times 3\tfrac{1}{3} = 9s.\ 4\tfrac{1}{2}d.$$

Work the example by algebra, putting x for D's share.

Ex. 5. *The sum of* £15. 10*s. is to be divided amongst* 3 *men,* 5 *women, and* 8 *boys, so that for every shilling a man gets, a woman gets* 8*d. and a boy* 6*d.: find the share of each.*

A woman's share $= \frac{2}{3}$ of a man's;

a boy's share $= \frac{1}{2}$ of a man's;

$\therefore$ a man's share + a woman's + a boy's $= \left(1 + \frac{2}{3} + \frac{1}{2}\right)$ of a man's;

$\therefore$ 3 men's shares + 5 women's + 8 boys' $= \left(3 + \frac{10}{3} + 4\right)$ of a man's
$= 10\frac{1}{3}$ times a man's share;

but the united shares of men, women, and boys = £$15\frac{1}{2}$;

$\therefore$ $10\frac{1}{3}$ times a man's share = £$15\frac{1}{2}$;

$$\therefore\ \text{a man's share} = £\frac{15\frac{1}{2}}{10\frac{1}{3}} = £1.\ 10s.;$$

and the other shares are easily found.

Work the example by algebra, putting x for a man's share; then $\frac{8}{12}x$ or $\frac{2}{3}x$ = a woman's share, and a boy's $= \frac{6}{12}x$ or $\frac{1}{2}x$, etc.

EXAMPLES. XXXVI.

1. Divide £35. 5*s.* $1\frac{1}{2}$*d.* among A, B, and C, so that B may get twice, and C 3 times as much as A.

2. Divide a legacy of £2375. 4*s.* 2*d.* among a son, a wife, and a daughter, so that the son's share shall be 3 times the wife's, and the wife's 3 times the daughter's.

3. Divide 4 guineas between A and B, so that $\frac{1}{2}$ of A's money shall be equal to $\frac{2}{3}$ of B's.

4. Divide £22. 9*s.* 9*d.* among 3 persons, so that the first may have twice as much as the second, and 3 times as much as the third.

5. Three soldiers, *A*, *B*, and *C*, divide 1155 cartridges among them: as often as *A* takes 4, *B* takes 3, and as often as *A* takes 6, *C* takes 7: find how many each has.

6. Divide 7 guineas between *A* and *B*, so that *B*'s share may be $\frac{3}{11}$ of *A*'s.

7. Divide 7872 into three such parts that the first shall be 4 times the second, and the second 3 times the third.

8. A man wishes to divide £11400 among his 2 sons and a daughter in such a way that the elder son may get half as much again as the younger, and the younger son half as much again as the daughter: find their shares.

9. Divide £180 among *A*, *B*, and *C*, so that *A* may receive 3 times as much as *B*, and *B* and *C* together $\frac{1}{2}$ as much as *A*.

10. Divide £64 among *A*, *B*, and *C*, so that *A* may have 3 times as much as *B*, and *C* $\frac{1}{3}$ of what *A* and *B* together have.

11. A sum of money was divided among 3 persons, *A*, *B*, and *C*: *A* and *B* together got £18. 19*s*., *A* and *C* together got £17. 13*s*. 6*d*., and *B* and *C* together got £15. 12*s*. 6*d*.: what was the sum divided, and how much did each receive?

12. £2000 is to be divided among *A*, *B*, and *C*, so that *B*'s share may be $\frac{7}{8}$ of *A*'s share, and *C*'s share $\frac{5}{6}$ of *B*'s: find each one's share.

13. If *A*, *B*, and *C* together have £2716. 15*s*. 3*d*., and *A*'s share $=\frac{2}{3}$ of *B*'s, and *C*'s $=\frac{1}{5}$ of *A*'s, find the share of each of them.

14. Divide £8424 among *A*, *B*, and *C*, so that *A* shall have $\frac{4}{9}$ as much as *B* and *C* together, and *B* $\frac{4}{5}$ of what *A* and *C* together receive.

15. *A* spends all his income, which is $\frac{2}{5}$ of *B*'s, and *B* spends as much as *A*, and saves £1 per week: find the income of each.

16. £405 is divided among *A*, *B*, *C*, and *D*: *A* gets £10 more than *C*, and *B* £15 more than *D*, and *A*'s share is equal to *D*'s: find the share of each.

17. Divide £9. 11*s*. 4*d*. among 10 men, 6 women, and 4 children, giving a woman 3 times as much as a child, and a man twice as much as a woman.

18. *A* and *B* together earn £3. 16*s*. in 8 days, *A* and *C* together earn £7. 13*s*. in 17 days, and *B* and *C* together £12. 15*s*. in 30 days: find how much a day they severally earn.

19. If 12 men, 10 women, and 22 children, working together, earn £16. 13*s*. in a week, and one woman earns as much as three children, and a man and a child as much as three women, what are the daily wages of each man, woman, and child?

20. Divide £2259. 10*s*. 7*d*. among four persons, giving to the second half as much again as to the first, to the third one-third less than to the first and second together, and to the fourth one-third less than to the second and third together.

CHAPTER VIII.

DECIMALS AND INDICES.

LET the accompanying figure represent a unit divided into 10 equal parts by the heavy lines and into 1000 equal parts by the light lines. Then each of the large squares is $\frac{1}{10}$ and each of the small squares $\frac{1}{1000}$ of the whole figure.

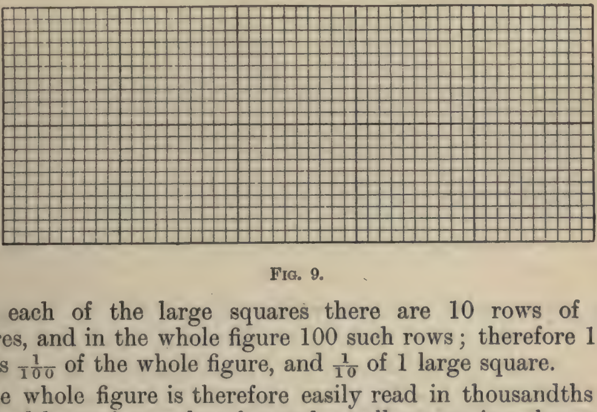


FIG. 9.

In each of the large squares there are 10 rows of small squares, and in the whole figure 100 such rows; therefore 1 such row is $\frac{1}{100}$ of the whole figure, and $\frac{1}{10}$ of 1 large square.

The whole figure is therefore easily read in thousandths or in hundredths or in tenths; for each small square is a thousandth, each row of small squares a hundredth, and each large square a tenth, of the whole figure, which is, of course, denoted by 1;

$$\therefore\ 1 = 10 \text{ tenths} = 100 \text{ hundredths} = 1000 \text{ thousandths}.$$

A study of the figure shows:

(*a*) that $\frac{1}{10}$ of it $= \frac{10}{100}$ of it $= \frac{100}{1000}$ of it;
$\frac{2}{10}$ of it $= \frac{20}{100}$ of it $= \frac{200}{1000}$ of it;
$\frac{5}{10}$ of it $= \frac{50}{100}$ of it $= \frac{500}{1000}$ of it;

(*b*) that a thousandth is $\frac{1}{10}$ of a hundredth, and a hundredth $\frac{1}{10}$ of a tenth.

In decimal notation we express:

Tenths.	Hundredths.	Thousandths.
$\frac{1}{10}$ as ·1,	$\frac{1}{100}$ as ·01,	$\frac{1}{1000}$ as ·001,
$\frac{2}{10}$ as ·2,	$\frac{2}{100}$ as ·02	$\frac{2}{1000}$ as ·002,
$\frac{5}{10}$ as ·5,	$\frac{5}{100}$ as ·05,	$\frac{5}{1000}$ as ·005.

The decimal point · is used to denote that the figures to the right of it represent parts of a unit, the first representing tenths of the unit, the second tenths of a tenth, that is, hundredths of the unit, the third tenths of a hundredth, that is, thousandths of the unit; and so on, each figure representing parts of $\frac{1}{10}$ of the value of those represented by the figure next to it on the left, just as in integral notation.

A study of the notation given above shows:

(1) That the number of places *after* the decimal point corresponds to the power of 10 constituting the denominator of the equivalent vulgar fractional form that the decimal form represents.

(2) That the last figure in the decimal occupies a place as many removes to the right of the decimal point as there are zeros in the denominator of the equivalent vulgar fractional form.

(3) That absent denominations (tenths in the second column, and tenths and hundredths in the third) are represented by 0's.

(4) That in decimal notation the denominator is not expressed.

Let it be required to express in decimal notation the value of 9, 35, *and* 147 *of the small squares of our figure.*

9 small squares $= \frac{9}{1000}$ of the whole figure.

The denominator $1000 = 10^3$, therefore the decimal must have 3 places after the decimal point; and, tenths and hundredths being absent, their places must be occupied by 0's.

Hence 9 small squares = ·009 of the whole figure.

35 small squares $= \frac{35}{1000}$ of the whole figure.

The denominator $1000 = 10^3$; therefore the decimal must have 3 places after the decimal point.

Reference to the figure shows that 35 thousandths = 3 hundredths + 5 thousandths. Tenths are absent, and therefore must be represented by a 0.

Hence 35 small squares = ·035 of the whole figure.

147 small squares = $\frac{147}{1000}$ of the whole figure.

The denominator being 10^3, the decimal must have 3 places after the decimal point.

Reference to the figure shows that 147 thousandths = 1 tenth + 4 hundredths + 7 thousandths. No denomination is absent;

∴ 147 small squares = ·147 of the whole figure.

·009 is read *point, 0, 0, 9* or *decimal, 0, 0, 9.*

·035 is read *point, 0, 3, 5* or *decimal, 0, 3, 5.*

·147 is read *point, 1, 4, 7* or *decimal, 1, 4, 7.*

(*a*) Represent the following numbers of small squares as decimals of the figure;

27; 85; 325; 768; and 951.

(*b*) Represent in small squares the values of the following decimals of the figure:

·067; ·005; ·379; ·653; ·952; and ·7.

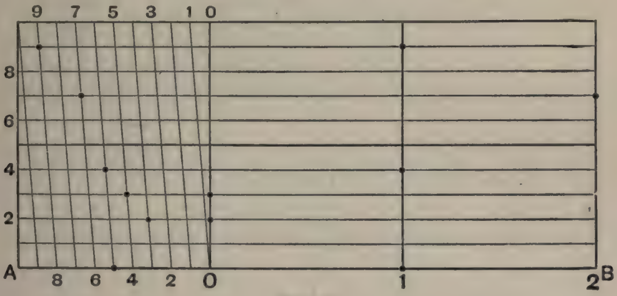


FIG. 10.

The figure given above is called a *diagonal scale.* The base line *AB* is 3 inches long, the left-hand inch of it being divided into tenths of an inch. We can therefore measure with this line to tenths of an inch any line not more than 3 inches long.

Parallel to the base line are drawn 10 other lines, each of which is cut by the diagonal lines drawn from the points 0, 1, 2 ... of the base to the points 1, 2, 3 ... of the top line.

The diagonal 01 and the vertical 00 cut off from the parallels above the base line the following parts:

from the 1st parallel $\frac{1}{10}$ of $\frac{1}{10}$ of an inch or ·01 of an inch,

from the 2nd parallel $\frac{2}{10}$ of $\frac{1}{10}$ of an inch or ·02 of an inch,

from the 3rd parallel $\frac{3}{10}$ of $\frac{1}{10}$ of an inch or ·03 of an inch,

and so on as we ascend the scale, the part cut off from each parallel being ·01 of an inch longer than the part cut off from the parallel next below it.

Hence with the parallels above the base line we can measure to hundredths of an inch any length up to 3 inches. For example, the lengths marked on the scale by the dots measure respectively:

Base - - - - - -	1·5 in.,
Second parallel above the base -	·32 in.,
Third parallel above the base - -	·43 in.,
Fourth parallel above the base -	1·54 in.,
Seventh parallel above the base -	2·67 in.,
Ninth parallel above the base - -	1·89 in.

Observe that the tenths' digit is found by following the diagonal to the base line, and that the hundredths' digit corresponds to the number of the parallel along which the measurement is made.

It is obvious that, if 1 inch-division on the scale is taken to represent 1 yard, then 2·67 inches on the scale will represent 2·67 yards; and that, if 1 inch-division on the scale is taken to represent 1 chain, 2·67 inches on it will represent 2·67 chains, that is, 2 chains 67 links.

1. Measure the following lines with the scale.

2. Draw lines of the following lengths:

2·78 inches, 1·03 inches, 2·68 inches.

Comparison of centimetres and millimetres with inches and tenths of an inch.

A = 1 decimetre = 10 centimetres = 100 millimetres.

B = 4 inches, the right-hand inch being divided into tenths of an inch.

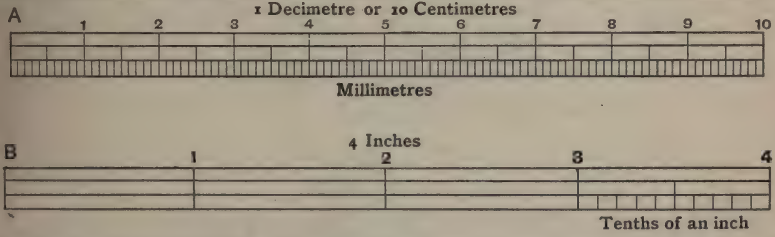
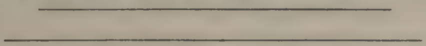


FIG. 11.

1. Measure the following lines with both of the above scales:



2. Find from the scales the ratio of a centimetre to an inch, and the ratio of an inch to a centimetre.

3. A man walks 2·7 miles due east and then 1·9 mile due south: draw a plan, representing 1 mile by 1 inch, and find by measurement his distance from his starting point.

4. A man walks 6·4 miles due west and then 5·8 miles due north: draw a plan, representing 1 mile by 1 cm., and find his distance from his starting point.

5. Draw squares on sides of the following lengths:

2·8 inches, 1·67 inches, 4·5 cm., ·45 dm.

Notation by Powers of 10.

We have already seen that $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, $10^4 = 10000$, and so on, where the *index* figure is *positive.*

We have now to assign a meaning to 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, and so on, where the *index* figure is *negative.*

$$10^{-1} = \frac{1}{10}; \quad 10^{-2} = \frac{1}{10^2}; \quad 10^{-3} = \frac{1}{10^3}; \quad 10^{-4} = \frac{1}{10^4}.$$

Hence
$$\begin{aligned}
75\times10^{-1}&=\frac{75}{10}=7{\cdot}5\,;\\
75\times10^{-2}&=\frac{75}{10^2}=\frac{75}{100}\quad=\cdot75\,;\\
75\times10^{-3}&=\frac{75}{10^3}=\frac{75}{1000}\quad=\cdot075\,;\\
75\times10^{-4}&=\frac{75}{10^4}=\frac{75}{10000}=\cdot0075.
\end{aligned}$$

The meaning of 0 as an index.

$$4^3\div4^3=4^{3-3}=4^0,$$

and $$4^3\div4^3=\tfrac{4\times4\times4}{4\times4\times4}=1.$$

$$10^4\div10^4=10^{4-4}=10^0,$$

and $$10^4\div10^4=\tfrac{10\times10\times10\times10}{10\times10\times10\times10}=1.$$

$$n^x\div n^x=n^{x-x}=n^0,$$

and $$n^x\div n^x=\frac{n\times n\times n\ldots\text{ to }x\text{ factors}}{n\times n\times n\ldots\text{ to }x\text{ factors}}=1.$$

Hence *a quantity with index* 0 *equals* 1.

The following table of powers of 10 will make clear the significance of index notation, positive and negative:

$$1=10^0,$$

$$\begin{array}{rr}
10=10^1, & \tfrac{1}{10}=10^{-1},\\
100=10^2, & \tfrac{1}{100}=10^{-2},\\
1000=10^3, & \tfrac{1}{1000}=10^{-3},\\
10000=10^4, & \tfrac{1}{10000}=10^{-4},\\
100000=10^5, & \tfrac{1}{100000}=10^{-5},\\
\text{and so on.} & \text{and so on.}
\end{array}$$

EXAMPLES. XXXVII.

Find the values of:

1. 6×10^{-1}; 16×10^{-1}; 265×10^{-1}.

2. 9×10^{-2}; 29×10^{-2}; 127×10^{-2}.

3. 8×10^{-3}; 87×10^{-3}; 875×10^{-3}.

4. 7×10^{-4}; 35×10^{-4}; 728×10^{-4}.

The product of powers of the same quantity.

We have already dealt with powers expressed by means of positive indices, and have now to consider the case of powers expressed by means of negative indices. We have seen that

$$10^2 \times 10^3 \times 10^4 = 10^{2+3+4} = 10^9,$$

and we have now to see that

$$10^{-2} \times 10^{-3} \times 10^{-4} = 10^{-(2+3+4)} = 10^{-9};$$

for $$10^{-2} \times 10^{-3} \times 10^{-4} = \frac{1}{10^2} \times \frac{1}{10^3} \times \frac{1}{10^4} = \frac{1}{10^9} = 10^{-9};$$

and, generally, $n^{-x} \times n^{-y} \times n^{-z} = n^{-(x+y+z)}$.

When the indices are mixed, some being positive and others negative, the difference of the indices is the index of the power representing the product. Thus,

$$10^{-2} \times 10^{-3} \times 10^4 = 10^{4-2-3} = 10^{-1} = \tfrac{1}{10} = \cdot 1.$$

The quotient of powers of the same quantity.

We have already shown that

$$5^4 \div 5^2 = 5^{4-2} = 5^2,$$

$$10^5 \div 10^2 = 10^{5-2} = 10^3, \text{ and so on};$$

that is, that the quotient of two powers of the same quantity is expressed by writing the quantity with an index that is the difference between the indices of the dividend and the divisor. The same method is adopted when the indices are negative, and also when one is negative and the other positive. Thus,

$$10^{-4} \div 10^{-2} = 10^{-4-(-2)} = 10^{-4+2} = 10^{-2};$$

for $$10^{-4} \div 10^{-2} = \frac{1}{10^4} \div \frac{1}{10^2} = \frac{1}{10^4} \times \frac{10^2}{1} = \frac{1}{10^2} = 10^{-2}.$$

$$7^3 \div 7^{-2} = 7^{3-(-2)} = 7^{3+2} = 7^5;$$

for $$7^3 \div 7^{-2} = 7^3 \div \frac{1}{7^2} = 7^3 \times \frac{7^2}{1} = 7^5;$$

and, generally, $a^m \div a^{-n} = a^{m-(-n)} = a^{m+n}$,

and $a^{-m} \div a^{-n} = a^{-m-(-n)} = a^{n-m}$.

From this is derived the following rule for such cases:

Change the sign of the index of the divisor, and add the index to that of the dividend.

The foregoing results are of great importance in the multiplication and division of decimals, and they constitute the ground-work of logarithmic computation.

To write in index notation a **mixed decimal,** *that is a decimal expression consisting of an integral and a decimal part, as* 4753·869.

The integral part is 4753, which, as we have seen, is in index notation written $4 \times 10^3 + 7 \times 10^2 + 5 \times 10^1 + 3$.

The decimal part is ·869, which is in index notation written

$$8 \times 10^{-1} + 6 \times 10^{-2} + 9 \times 10^{-3},$$

$$\text{meaning} \quad \frac{8}{10} + \frac{6}{10^2} + \frac{9}{10^3},$$

$$\text{that is,} \quad \frac{8}{10} + \frac{6}{100} + \frac{9}{1000}.$$

$$\text{Hence} \quad 4753{\cdot}869$$

$$= 4 \times 10^3 + 7 \times 10^2 + 5 \times 10^1 + 3 + 8 \times 10^{-1} + 6 \times 10^{-2} + 9 \times 10^{-3},$$

an expression that shows the units' place to be the centre of symmetry of our system of notation, the *first* digit to the *left* of it representing so many times 10^1 and the *first* digit to the *right* of it so many times 10^{-1}, the *second* digit to the *left* of it representing so many times 10^2 and the *second* digit to the *right* of it so many times 10^{-2}, the *third* digit to the *left* of it representing so many times 10^3 and the *third* digit to the *right* of it so many times 10^{-3}.

The decimal point is therefore a kind of suffix, indicating that the digits to the left of it represent *tens, hundreds, thousands*, and so on, and that the digits occupying corresponding places to the right of it represent *tenths, hundredths, thousandths*, and so on.

Hence digits to the left of the units' place are associated with *multiplication* by a power of 10, and digits to the right of it with *division* by a power of 10, the former therefore increasing in value as they move to the left and the latter decreasing in value as they move to the right.

In counting places, whether to right or left, we begin with the units' place as *zero*; for $3 \times 10^0 = 3$, the notation 10^0 denoting 1.

It is often necessary to consider a number from the point of view of its order of magnitude, that is, the power of 10 which it represents. The numbers 7, 9, 3 are of the order 0, for they denote 7×10^0, 9×10^0, 3×10^0; the numbers 69, 78, 50 are of the order 1, for the 6, 7, 5 of these numbers denote 6×10^1, 7×10^1, 5×10^1; the numbers 725, 829, 900 are of the order 2, for the 7, 8, 9 of these numbers denote 7×10^2, 8×10^2, 9×10^2; the numbers 4728, 2631, 1000 are of the order 3, for the 4, 2, 1 of these numbers denote 3×10^3, 2×10^3, 1×10^3; the numbers ·9, ·7, ·6 are of the order -1, for they denote 9×10^{-1}, 7×10^{-1}, 6×10^{-1}; the numbers ·07, ·05, ·04 are of the order -2, for they denote 7×10^{-2}, 5×10^{-2}, 4×10^{-2}.

The order of every digit of a number depends solely on the position of the digit with regard to the units' place, orders to the left being *plus* orders and orders to the right being *minus* orders. The minus order is sometimes denoted thus: $\bar{2}$, $\bar{3}$ instead of -2, -3.

When we say that the order of a digit is 3, we mean that the digit stands 3 places to the left of the units' place, and that its value is 3×10^3 or 3000; and, when we say that its order is $\bar{3}$, we mean that it stands 3 places to the right of the units' place, and that its value is 3×10^{-3}, that is,

$$\frac{3}{10^3} \text{ or } \frac{3}{1000} \text{ or } \cdot 003.$$

It follows from what has been said that, when a number has been written down and its units' figure marked, its value is not altered by the placing of 0's either before or after it. Thus:

$$\begin{aligned} 375\cdot 06 &= 0375\cdot 060 \\ &= 00375\cdot 0600 \\ &= 000375\cdot 06000; \end{aligned}$$

for the 0's placed on the left in the last line represent respectively 0×10^3, 0×10^4, 0×10^5, each of which $= 0$; and the 0's placed on the right represent respectively

$$\frac{0}{10^3}, \frac{0}{10^4}, \frac{0}{10^5}, \text{ each of which} = 0.$$

The orders of the digits are not changed, and therefore the value of the number is not changed.

On the other hand, 0's placed between the decimal point and the first significant figure after the decimal point do alter the value of the decimal. For example, in ·075 the order of 7 is $\bar{2}$, and the value $\frac{7}{10^2}$; in ·0075 the order of 7 is $\bar{3}$, and the value $\frac{7}{10^3}$, which is $\frac{1}{10}$ of ·075; in ·00075 the order of 7 is $\bar{4}$, and the value $\frac{7}{10^4}$, which is $\frac{1}{10}$ of ·0075, and so on, *every* 0 *placed between the decimal point and the first significant figure of the decimal reducing the decimal to* $\frac{1}{10}$ *of its former value.*

The same result is shown by an analysis of the decimal into its fractional equivalents, thus:

$$\cdot075 = \tfrac{0}{10} + \tfrac{7}{100} + \tfrac{5}{1000};$$
$$\cdot0075 = \tfrac{0}{10} + \tfrac{0}{100} + \tfrac{7}{1000} + \tfrac{5}{10000};$$

and so on.

Consider the values of the following expressions:

$$\begin{aligned} 777\cdot &= 700 + 70 + 7; \\ 77\cdot7 &= 70 + 7 + \tfrac{7}{10} = \tfrac{1}{10} \text{ of } (700 + 70 + 7); \\ 7\cdot77 &= 7 + \tfrac{7}{10} + \tfrac{7}{100} = \tfrac{1}{10} \text{ of } (70 + 7 + \tfrac{7}{10}); \\ \cdot777 &= \tfrac{7}{10} + \tfrac{7}{100} + \tfrac{7}{1000} = \tfrac{1}{10} \text{ of } (7 + \tfrac{7}{10} + \tfrac{7}{100}). \end{aligned}$$

Or thus:

$$\begin{aligned} 777\cdot &= \tfrac{777}{1}; \\ 77\cdot7 &= \tfrac{777}{10} = \tfrac{1}{10} \text{ of } \tfrac{777}{1}; \\ 7\cdot77 &= \tfrac{777}{100} = \tfrac{1}{10} \text{ of } \tfrac{777}{10}; \\ \cdot777 &= \tfrac{777}{1000} = \tfrac{1}{10} \text{ of } \tfrac{777}{100}. \end{aligned}$$

Hence shifting the decimal point 1, 2, 3 ... *places to the left is equivalent to dividing a number by* 10, 100, 1000 ... *respectively.*

The student can prove for himself *that shifting the decimal point* 1, 2, 3 ... *places to the right is equivalent to multiplying a number by* 10, 100, 1000 ... *respectively.*

That is, *lowering the order of a number by* 1, 2, 3 ... *is equivalent to dividing the number by* 10, 100, 1000 ... *respectively, and raising the order of a number by* 1, 2, 3 ... *is equivalent to multiplying the number by* 10, 100, 1000 ... *respectively.*

We have seen that any decimal can be expressed in the form of a vulgar fraction. We have now to see how to express in decimal form the value of a vulgar fraction.

Since $1=10$ tenths;

$$\therefore \frac{1}{2}=\frac{10\text{ tenths}}{2}$$

$=5$ tenths

$=\cdot 5.$

The form the working takes in practice:

$$\begin{array}{r} 2\,)\,\underline{1\cdot\;} \\ \cdot 5 \end{array}$$

0 mentally supplied after the decimal point.

Since $3=30$ tenths

$=300$ hundredths;

$$\therefore \frac{3}{4}=\frac{300\text{ hundredths}}{4}$$

$=75$ hundredths

$=\cdot 75.$

$$\begin{array}{r} 4\,)\,\underline{3\cdot\;\;} \\ \cdot 75 \end{array}$$

00 mentally supplied after the decimal point.

Since $5=50$ tenths

$=500$ hundredths

$=5000$ thousandths;

$$\therefore \frac{5}{8}=\frac{5000\text{ thousandths}}{8}$$

$=625$ thousandths

$=\cdot 625.$

$$\begin{array}{r} 8\,)\,\underline{5\cdot\;\;\;} \\ \cdot 625 \end{array}$$

000 mentally supplied after the decimal point.

Since $15=150000$ ten thousandths;

$$\therefore \frac{15}{16}=\frac{150000\text{ ten thousandths}}{16}$$

$=9375$ ten thousandths

$=\cdot 9375.$

$$\begin{array}{r} 16\,)\,\underline{15\cdot\;\;\;\;} \\ \cdot 9375 \end{array}$$

0000 mentally supplied after the decimal point.

Or thus:
$$\begin{array}{r} 4\,)\,\underline{15\cdot\;\;\;\;} \\ 4\,)\,\underline{3\cdot 75\;\;} \\ \cdot 9375 \end{array}$$

Observe in connection with these examples (1) that the vulgar fractions are expressed in their lowest terms, (2) that we virtually continue to affix zeros to the numerator until the denominator will divide into it without leaving a remainder, (3) that there are as many decimal places in the quotient as there are noughts affixed to the numerator; and (4) that the number of decimal places in the quotient corresponds to the power of 2 constituting the denominator of the vulgar fraction:

$2=2^1$: there is one decimal place.

$4=2^2$: there are two decimal places.

$8=2^3$: there are three decimal places.

$16=2^4$: there are four decimal places.

The student should prove that the same law holds when the denominator is some power of 5 or the product of some power of 2 and 5.

By affixing noughts to the numerator we multiply it by some power of 10; and, since the only prime factors of powers of 10 are 2 and 5, it follows that no vulgar fraction in its lowest terms, the prime factors of the denominator of which are not 2 or 5, or 2 and 5, can be expressed accurately as a decimal; for there will always be a remainder however far we carry the division. Let it be required, for example, to convert $\frac{7}{12}$ into its equivalent decimal

$$12 = 2 \times 2 \times 3.$$

The denominator contains the factor 3, which will not divide an exact number of times into any power of 10. Let it be tried.

$$12\,\underline{)\,700000}\atop{\cdot 58333}\text{, and so on;}$$

the remainder being always 4, however far we carry out the division.

EXAMPLES. XXXVIII.

Convert into equivalent decimals:

1. $\frac{5}{8}$.
2. $\frac{15}{16}$.
3. $7\frac{19}{32}$.
4. $\frac{3}{800}$.
5. $\frac{39}{40}$.
6. $\frac{7}{125}$.
7. $\frac{1}{512}$.
8. $5\frac{53}{64}$.
9. $7\frac{3}{200}$.
10. $\frac{49}{625}$.
11. $1\frac{5}{12}$ of $2\frac{5}{8}$.
12. $3\frac{1}{2} \div 2\frac{2}{3}$.
13. $1\frac{3}{4} - \frac{7}{8}$ of $\frac{18}{28}$.
14. $\dfrac{2\frac{1}{5}}{12\frac{8}{10}}$.
15. Reduce 14*s*. 7½*d*. to the decimal of £5.
16. What decimal of £1 is 8·4 of a penny?
17. Express ·35 of £1 in decimal terms of £2. 2*s*. 8*d*.
18. What decimal of a guinea is 6*s*. 6¾*d*.?
19. Reduce 1 ton 3 cwts. 14 lbs. to the decimal of 5 tons.
20. Express 7 ozs. as the decimal of 2 cwt.

In the metric system of weights and measures, the *base* or *root* is 10, and therefore what we have said about decimals applies exactly to numbers expressing metric measures. Examine a metre measure; you see that it is divided into 10 large equal parts called decimetres, each of which is divided into 10 equal parts called centimetres, and that each centimetre is divided into 10 equal parts called millimetres.

Hence

1 metre = 10 decimetres = 100 centimetres = 1000 millimetres;

∴ 1 decimetre = ·1 of a metre (order $\bar{1}$);
1 centimetre = ·01 of a metre (order $\bar{2}$);
1 millimetre = ·001 of a metre (order $\bar{3}$).

The metre has therefore the same relation to its submultiples that the units' figure has to the digits following the decimal point of a decimal. Also its relation to its multiples, the decametre, hectometre, and kilometre, is the same as that of the units' digit to the digits to the left of it in the integral part of a decimal expression; for

4379·685 metres

$$=(4\times10^3+3\times10^2+7\times10^1+9+6\times10^{-1}+8\times10^{-2}+5\times10^{-3}) \text{ metres}$$

$$=(4000+300+70+9+\tfrac{6}{10}+\tfrac{8}{100}+\tfrac{5}{1000}) \text{ metres,}$$

where the 4 is seen to be of the order 3, the 3 of the order 2, the 7 of the order 1, the 6 of the order $\bar{1}$, the 8 of the order $\bar{2}$, and the 5 of the order $\bar{3}$.

If we express 4379·685 metres in terms of decametres, we have 437·9685 decametres.

The 4 is now of the order 2, for the decimal point shows that the units' figure is 7. The order of the 5 is now $\bar{4}$.

If we express 437·9685 decametres in terms of hectometres, we have 43·79685 hectometres.

The 4 is now of the order 1, and the 5 of the order $\bar{5}$.

If we express 43·79685 hectometres in terms of kilometres, we have 4·379685 kilometres.

The order of the 4 is now 0, and the order of the 5 is $\bar{6}$.

As by shifting the decimal point to the left we have expressed the metres in terms of higher denominations, so by shifting the point to the right we can express the metres in terms of lower denominations.

Thus: 4379·685 metres = 43796·85 decimetres
= 437968·5 centimetres
= 4379685 millimetres,

where, in the last, the 4 is of the order 6, and the 5 of the order 0.

In the same way the value of any decimal expression can be expressed in terms of higher or of lower denominations.

Higher denominations:

4379·685 = 437·9685 tens
= 43·79685 hundreds
= 4·379685 thousands
= ·4379685 ten-thousands, and so on.

Lower denominations:

4379·685 = 43796·85 tenths
= 437968·5 hundredths
= 4379685 thousandths
= 43796850 ten-thousandths, and so on.

The aim of the foregoing treatment of decimals is to show that the notation of decimal fractions is simply an extension of the notation of integral numbers. It is an easy inference that the operations of addition, subtraction, multiplication, and division are performed with decimal expressions in precisely the same way as with integral expressions. The only new thing in decimals is the point marking the position of the units' place, that is, the place from which begin the ascending or *plus* orders on the left of it and the descending or *minus* orders on the right of it. The important thing to mind is the decimal point, which must be associated, not with the figure following it, but with the figure preceding it, namely, the units' figure. *Keep the points under each other and the digits expressive of the same denominations or orders of the same unit under each other, i.e., tens under tens, units under units, tenths under tenths, hundredths under hundredths, etc., and the working and answers will take care of themselves.*

Addition and Subtraction.

Ex. 1. *Find the sum of* 3·7, 28·07, ·968, ·0035, 350.

3·7000
28·0700
·9680
·0035
350·
382·7415

Ex. 2. *Find the difference between* 25·6 *and* 7·0865.

25·6000
7·0865
18·5135

There is no necessity for putting 0's for absent orders on the right of a decimal.

Ex. 3. *Find the sum of* 7·5 m., 57 cm., ·078 m., 25·065 m., 3·87 km.

$$\begin{array}{lrl}
(a) & 7\cdot5 & \text{m.} \\
 & \cdot57 & \text{m.} \\
 & \cdot078 & \text{m.} \\
 & 25\cdot065 & \text{m.} \\
 & \underline{3870\cdot} & \text{m.} \\
 & 3903\cdot213 & \text{metres.}
\end{array}$$

Or thus:

$$\begin{array}{lccccccc}
 & \text{Km.} & \text{hm.} & \text{Dm.} & \text{m.} & \text{dm.} & \text{cm.} & \text{m.} \\
(b) & & & & 7 & 5 & 0 & 0 \\
 & & & & & 5 & 7 & 0 \\
 & & & & & & 7 & 8 \\
 & & & 2 & 5 & 0 & 6 & 5 \\
 & 3 & 8 & 7 & 0 & 0 & 0 & 0 \\
\hline
 & 3 & 9 & 0 & 3 & 2 & 1 & 3
\end{array}$$

The (*b*) method is not recommended.

Ex. 4. *Find the difference between* 9 Dg. 7 dg. 8 mg. *and* 3·75 g.

$$\begin{array}{lrl}
(a) & 90\cdot708 & \text{g.} \\
 & \underline{3\cdot750} & \text{g.} \\
 & 86\cdot958 & \text{g.}
\end{array}$$

$$\begin{array}{lccccc}
 & \text{Dg.} & \text{g.} & \text{dg.} & \text{cg.} & \text{mg.} \\
(b) & 9 & 0 & 7 & 0 & 8 \\
 & & 3 & 7 & 5 & 0 \\
\hline
 & 8 & 6 & 9 & 5 & 8
\end{array}$$

The (*b*) method is not recommended.

EXAMPLES. XXXIX.

Find the sum of:

1. 100·1, ·0001, 125·125, 500.

2. ·172, ·06, 1·0004, 20·02, ·001.

3. 12·4873, 124·873, 1248·73, 12487·3, 1·24873, ·124873.

4. 270, ·072, ·0001, 100·3508, 5·7764, ·12345.

5. 3·75, ·0074, ·01, 47·1, ·000075, 6·074, ·0017.

6. ·5, ·0039, 74·25, ·00704, ·000009.

7. 25, 2·5, ·25, ·025, ·0025, ·00025, ·000025, ·0000025.

8. 100, ·1, ·01, ·001, ·0001, ·00001, ·000001, ·0000001.

9. 72·035 m., ·973 m., 963·75 m., 90·37 m., 7·09 m.

10. 5·27 g., ·719 g., 79·68 g., 698·078 g., ·275 g.

11. 28·672 l., 7·039 l., 5796·857 l., 40·76 l., ·083 l.

12. 27·3 f., ·75 c., 237·9 f., 25 c., 12·7 f.

13. Express 385903 mm. in terms of decametres.

14. Express 604727 l. in terms of hectolitres.

15. Multiply 37·842 by 10 and by 1000.

16. Divide 59486 hectometres by 100 and by 1000.

EXAMPLES. XL.

Find the difference between :

1. 1 and ·01.
2. 75 and ·075.
3. 23·5 and ·9876.
4. 17 and ·0017.
5. 172·8 and 17·28.
6. 1·44 and ·00144.
7. 6·4 and ·000064.
8. 480·5 and 4·805.
9. ·1 and ·00021.
10. ·001 and ·0001.
11. What number subtracted from 13·007 leaves 3·594 ?
12. What number subtracted from 1 leaves ·01024 ?
13. What number added to 13·265 makes up 100·0008 ?
14. What number added to ·0976 makes up 1·28 ?
15. From 72·354 m. take 19·76 m.
16. From 19 francs 2 centimes take 9·75 francs.
17. From 3 hl. 6 Dl. 2 l. 3 dl. 4 cl. take 79·753 l.
18. From 232·35 g. take 7 Dg. 8 g. 7 dg. 8 cg. 5 mg.

Multiplication.

(*a*) By a whole number.

Ex. 1. *Find the value of* 72·34 × 28.

$$\begin{array}{r}72{\cdot}34\\ 28{\cdot}\\ \hline 1446{\cdot}8\\ 578{\cdot}72\\ \hline 2025{\cdot}52\end{array}$$

We first multiply by 2 (tens). The first right-hand figure in the first partial product is 8, for $\frac{4}{100}\times 20=\frac{8}{10}={\cdot}8$. The 8 therefore fixes the places of the rest of the digits of the first partial product.

We next multiply by 8. The first right-hand figure of the second partial product is 2, for $\frac{4}{100}\times 8=\frac{32}{100}=\frac{3}{10}+\frac{2}{100}$. The 2 therefore fixes the places of the rest of the digits in the second partial product.

Or thus, the point in the multiplicand being for the moment disregarded, and the multiplication beginning on the right :

$$\begin{array}{r}72{\cdot}34\\ 28\\ \hline 57872\\ 14468\\ \hline 202552\end{array}$$

Here the multiplicand has had assigned to it a value 100 times as great as its real value ; therefore the product is 100 times the value of the true product. Hence to get its true value, we divide 202552 by 100, that is, we point off two figures from the right, thus : 2025·52.

The disadvantage of this method is that it does not show clearly the value of each of the partial products ; but, as partial products are of no great importance in actual computation, no serious evil results from its employment.

Again, $$72{\cdot}34=7234\times 10^{-2}=\frac{7234}{100};$$

$$\therefore\ 72{\cdot}34\times 28=\frac{7234}{100}\times 28=\frac{202552}{100}=2025{\cdot}52,$$

where we have treated the multiplicand and multiplier as whole numbers and divided the product by the power of 10 denoted by the decimal in the actual multiplicand.

(*b*) A decimal multiplied by a decimal.

Ex. 2. *Find the value of* 52·683 × 46·342.

$$\begin{array}{r}
52{\cdot}683\\
46{\cdot}342\\
\hline
2107{\cdot}32\\
316{\cdot}098\\
15{\cdot}8049\\
2{\cdot}10732\\
{\cdot}105366\\
\hline
2441{\cdot}435586
\end{array}$$

Work showing how the place of the right-hand figure in each partial product is determined:

(1) $4\times10^{1}\ \ \times3\times10^{-3}=4\times3\times10^{-2}=\frac{12}{100}$.

(2) $6\times10^{0}\ \ \times3\times10^{-3}=6\times3\times10^{-3}=\frac{18}{1000}$.

(3) $3\times10^{-1}\times3\times10^{-3}=3\times3\times10^{-4}=\frac{9}{10000}$.

(4) $4\times10^{-2}\times3\times10^{-3}=4\times3\times10^{-5}=\frac{12}{100000}$.

(5) $2\times10^{-3}\times3\times10^{-3}=2\times3\times10^{-6}=\frac{6}{1000000}$.

Again, since
$$\begin{aligned}52{\cdot}683\times46{\cdot}342&=52683\times10^{-3}\times46342\times10^{-3}\\&=52683\times46342\times10^{-6},\end{aligned}$$

we may multiply as with whole numbers and divide the number by the power of 10 indicated by the negative index 6.

Beginning with the highest digit in the multiplier:

$$\begin{array}{r}
52683\\
46342\\
\hline
210732\\
316098\\
158049\\
210732\\
105366\\
\hline
2441435586
\end{array}$$

Beginning with the lowest digit in the multiplier:

$$\begin{array}{r}
52683\\
46342\\
\hline
105366\\
210732\\
158049\\
316098\\
210732\\
\hline
2441435586
\end{array}$$

Dividing the product by 10^6, that is, 1000000, we have 2441·435586.

Hence we may apply the method of multiplication of integers to the multiplication of decimals if we *mark off in the product a number of decimal places equal to the sum of the number of decimal places contained in the multiplicand and multiplier together.*

Ex. 3. *Find the product of* ·00027 *and* ·26

$$\begin{aligned}{\cdot}00027\times{\cdot}26&=27\times10^{-5}\times26\times10^{-2}\\&=27\times26\times10^{-7};\end{aligned}$$

∴ the product will be a decimal of the order $\overline{7}$, that is, it will contain 7 decimal places.

$$\begin{array}{r}
27\\
26\\
\hline
162\\
54\\
\hline
{\cdot}0000702
\end{array}$$

Since the product of 27 and 26 does not give 7 figures, we prefix 0's to make up the number.

Consideration of the orders of the digits multiplied affords a useful test of the accuracy of the placement of results, and also a rough test of the accuracy of the orders of the digits in the final result. For example:

7 (of the order 2) × 6 (of the order 1) = 42 (of the order 3)
= 42 thousands, or 42000.

6 (of the order 3) × 5 (of the order $\bar{1}$) = 30 (of the order 2)
= 30 hundreds, or 3000.

7 (of the order $\bar{2}$) × 8 (of the order $\bar{3}$) = 56 (of the order $\bar{5}$)
= 56 hundred-thousandths,
that is, ·00056.

Examples.

(1) $$7{\cdot}35 \times {\cdot}003 = 735 \times 10^{-2} \times 3 \times 10^{-3}$$
$$= 735 \times 3 \times 10^{-5};$$

∴ the answer must contain 5 decimal places; and, since the 7 of 7·35 is of the order 0, and the 3 of ·003 is of the order $\bar{3}$, 7 × 3 or 21 is of the order − 3, that is, ·021, which is a *rough approximation* to the product of 7·35 and ·003.

(2) $$73{\cdot}864 \times 395{\cdot}42 = 73864 \times 10^{-3} \times 39542 \times 10^{-2}$$
$$= 73864 \times 39542 \times 10^{-5};$$

∴ the decimal part of the answer will contain five places;

and, since the 7 of 73·864 is 7×10^1,

and the 3 of 395·42 is 3×10^2;

∴ 7 × 3 or 21 is of the order 3, that is, 21000, which is a *rough approximation* to the product of 73·864 and 395·42. It is, of course, an underestimate, for it does not take account of the digits that follow 7 and 3.

The habit of making rough approximations to the results sought will save the student from blundering into absurd answers.

The accompanying rectangular figure is 3·6 inches long and 2·3 inches broad. Each large square is a square inch, and each small square ·01 of a square inch. By counting we find that there are 6 large squares and 228 small squares. Since 100 of the latter make 1 of the former, 228 of the small squares make 2·28 of the large ones, that is 2·28 square inches. Hence the total area of the figure is 8·28 square inches.

We may, of course, let the length of the side of the large square represent any unit, for example, a yard (scale 1 inch to the yard); then the figure would represent 8·28 square yards. Suppose the side of the large square to represent a chain (scale 1 inch to the chain); then the figure would represent 8·28 square chains; and so on, with other units.

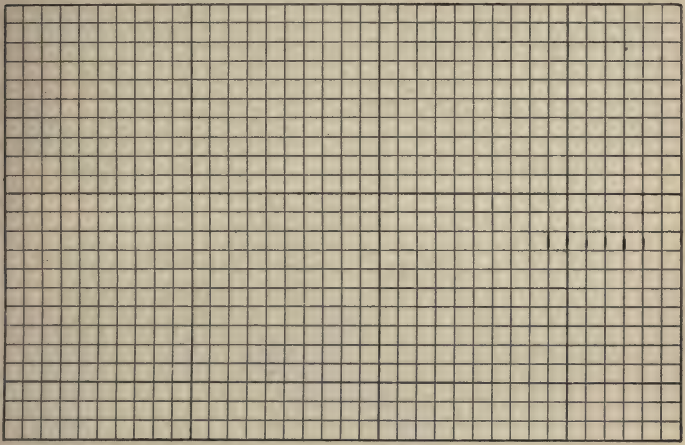


FIG. 12.

We get the same result if we multiply the number of linear units in the length by the number of linear units in the breadth of the rectangular figure, thus:

$$\begin{array}{l|r} \text{Area} = l \times b & 3{\cdot}6 \\ \phantom{\text{Area}} = 3{\cdot}6 \times 2{\cdot}3 & \underline{2{\cdot}3} \\ \phantom{\text{Area}} = 8{\cdot}28 \text{ square units.} & 10\,8 \\ & \underline{72\;} \\ & 8{\cdot}28 \end{array}$$

EXAMPLES. XLI.

1. Multiply 25·705 m. by 45, and ·068 m. by 29.

2. Express in hectograms the product of 125·206 g. by 81.

3. Express in kilograms the product of 2305·87 g. by 19.

4. How many decalitres in 18 times 35·037 litres?

5. How many kilolitres in 27 times 4 hl. 7 l. 6 dl. 9 ml.?

6. Find the value of 15·75 m. of planking at 2·3 f. per metre.
7. Find the cost of 27·35 l. of wine at 2·5 f. per litre.
8. How many centimetres in x hectometres?
9. How many centimes in y francs?
10. Find the value of y decalitres at x francs per litre.

Draw on squared paper, and find the area of rectangular surfaces of the following dimensions:

11. 3·4 yards by 4·7 yards (scale 1 inch to the yard).
12. 5·8 metres by 4·6 metres (scale 1 cm. to the metre).
13. 4·8 decametres by 4·8 decametres (scale 1 cm. to the decametre).

Find the values of:

14. 32·25 × 1·004.
15. 24·35 × ·074.
16. 1703·02 × 93·05.
17. 34·742 × 3·25.
18. 7·045 × 974·01.
19. ·0057 × 9·94.
20. 2·345 × ·0024.
21. 6 × ·0005.
22. 57·013 × 00029.
23. ·02057 × ·0039.
24. ·0125 × 20·08.
25. 1·00025 × 2400.

Division.

Ex. 1. *Divide* ·0646 *by* 17.

When the divisor is a whole number, we generally proceed as follows:

Quotient : ·0038
17) ·0646
136
0

Or thus :
17) ·0646 (·0038
136
0

Ex. 2. *Divide* ·0646 *by* ·17.

$$\cdot0646 \div \cdot17 = 646 \times 10^{-4} \div 17 \times 10^{-2}$$
$$= \frac{646 \times 10^{-4}}{17 \times 10^{-2}} = \frac{646 \times 10^{-2}}{17}$$
$$= \frac{6\cdot46}{17} = \cdot38.$$

The stage $\frac{646 \times 10^{-2}}{17}$ shows that the quotient is a decimal of the order $\overline{2}$, and that its first digit is of the order $\overline{1}$.

It is always easy to determine at sight the order of the first significant figure of the quotient. In the last example, for instance, the first step in the division is $\frac{64}{17}$. The order of the 64 is -3 or $\overline{3}$, and the order of the 17 is -2 or $\overline{2}$;

$\therefore$ order of dividend − order of divisor $= -3-(-2)$
$= -3+2 = -1$;

$\therefore$ the first significant figure in the quotient is tenths.

Ex. 3. *Divide* 526·7 *by* ·0229.

We shall treat the dividend and the divisor as if they were whole numbers, and then proceed to determine the value of the quotient.

```
229 ) 5267 ( 23
       687
         0
```

The first step is $\frac{526}{229}$; the order of the 526 (considered as units) is 0, and the order of the 229 is -4;

$\therefore$ the order of the first significant figure of the quotient is 4;

that is, the value of the 2 in quotient $= 2 \times 10^4 = 20000$ units;

$\therefore$ the value of the quotient is 23000.

Otherwise:

$$526{\cdot}7 \div {\cdot}0229 = 5267 \times 10^{-1} \div 229 \times 10^{-4}$$
$$= \frac{5267 \times 10^{-1}}{229 \times 10^{-4}}$$
$$= \frac{5267}{229 \times 10^{-3}}$$
$$= \frac{5267 \times 10^3}{229}$$
$$= 23 \times 10^3 = 23000.$$

From the working of the examples, we derive the following rule:

Divide as in whole numbers, and determine the value of the first significant figure of the quotient by subtracting the order of the divisor from the order of that part of the dividend into which the divisor is first divided.

Or the following method may be adopted.

Ex. 1. *Find the quotient of* 1748 *divided by* ·38.

The dividend is an integral number; and the divisor is a decimal of the second minus order. Express the dividend as a decimal of the second minus order.

1748 = 1748·00 = 174800 hundredths.

It is required to find how many times 38 hundredths are contained in 174800 hundredths. This is plainly identical in principle with the question: How many groups of 38 eggs are contained in 174800 eggs? It is in fact a question in simple division.

```
38 ) 174800 ( 4600
      228
        0
```

$\therefore$ 1748 ÷ ·38 = 4600.

Ex. 2. *Find the quotient of* 17·28 *divided by* ·0012.

Express the dividend as a decimal of the fourth minus order.

17·28 = 17·2800 = 172800 ten-thousandths.

It is required to find how many times 12 ten-thousandths are contained in 172800 ten-thousandths. This is identical in principle with the concrete case cited above; hence it is a question in simple division.

12) 172800
14400

∴ 17·28 ÷ 0012 = 14400.

Ex. 3. *Find the value of* 347 ÷ ·64.

The dividend expressed as a decimal of the second minus order is 347·00; that is, 34700 hundredths. We have to find how often 64 hundredths are contained in 34700 hundredths, which is plainly a question in simple division.

64) 34700 (542·1875
270
140
12·0
5·60
·480
·0320
0

∴ 347 ÷ ·64 = 542·1875.

The decimal point is put here only to show the values of the partial remainders and to indicate where the decimal part of the quotient begins.

Ex. 4. The area of a rectangular floor is 18·33 square metres, and the length 4·7 metres: find the breadth.

4·7 metres = 4·70 metres.

$47\cancel{0}$) 1833 (3·9
423
0

By the long method:

$47\cancel{0}$) 1833 (3·9
1410
423
423

Dividing the divisor by 10 is equivalent to multiplying the dividend by 10, and therefore, to obtain the tenths' digit of the quotient, we have deleted the 0 of the divisor instead of affixing a 0 to the dividend.

From the foregoing examples it appears that *the operation of dividing decimals of the same order into each other is identical with dividing whole numbers into each other.* Hence we may give the following rule for the division of decimal expressions into each other:

Express the dividend and the divisor as decimals of the same order, and proceed as in division of whole numbers.

EXAMPLES. XLII.

1. Divide 6 dm. 5 m. 2 dm. 5 cm. by 9.

2. Divide 476·064 grams by 12.

3. Divide 3395·925 litres by 75.

4. Find a thirty-ninth of 3161·964 grams.

5. Divide 340 m. 7 dm. 5 cm. by 47.

6. How many measures 27·25 m. long are contained in 1253·5 m.?

7. A rectangular floor contains 340·6875 square metres, and its width is 17·25 metres: find its length.

8. If a boy's step is ·625 m., how many steps will he take in 46·875 metres?

9. How many measures of 735·5 litres are contained in 25·7425 kilolitres?

10. A farmer buys lambs at £·75 each: if he spends £43·5, how many lambs does he buy?

11. 24·1 ÷ ·0025.
12. 750·35 ÷ ·000215.
13. 585·98 ÷ ·083.
14. 56·28 ÷ ·2345.
15. 17·25 ÷ ·0023.
16. 3075 ÷ ·125.
17. 312 ÷ ·0325.
18. 3·375 ÷ ·00005.
19. 84·375 ÷ ·00375.
20. 72184401 ÷ 88·353.
21. 30·5118 ÷ 50·6.
22. 21·97 ÷ 1690.
23. 21·97 ÷ 1·69.
24. 21·97 ÷ ·169.
25. ·144 ÷ 1200.
26. 3·12 ÷ 325.
27. 1·69 ÷ ·013.
28. ·002 ÷ 1·6.
29. 3·375 ÷ 250.
30. 3·375 ÷ 2·25.

31. Find the average of 2·76, 7·35, 8·75, 18·06, 3·37, 6·29, 10·08, 7·83.

32. The average of 8 numbers is 4·65625, and the average of half the numbers is 3·2125: find the average of the other half.

CHAPTER IX.

APPROXIMATION AND DEGREES OF ACCURACY.

Working to specified degrees of accuracy.

Let it be required to find the value of 3·7638 "correct" (1) to *one* decimal place, (2) to *two* decimal places, (3) to *three* decimal places.

(1) If we take 3·7 as the value of 3·7638, we take too little by ·0638, and if we take 3·8, we take too much by ·0362. The error in the second case is therefore less than that in the first. Hence we take 3·8 as a closer approximation to the value of 3·7638 than is 3·7.

(2) If we take 3·76 as the value of 3·7638, the error is ·0038, and if we take 3·77, the error is ·0062. Hence the error in the second case is greater than that in the first. We therefore take 3·76 as the value that approximates the more closely to the value of 3·7638.

(3) If we take 3·763 as the value of 3·7638, the error is ·0008, and if we take 3·764, the error is ·0002. The error in the second case is less than that in the first. Hence we take 3·764 as a closer approximation to the value of 3·7638 than is 3·763.

Hence the value of 3·7638 "correct to 1 decimal place" is 3·77, "correct to 2 decimal places" is 3·76, and "correct to 3 decimal places" is 3·764.

The expression "correct to 1, 2, 3 ... decimal places" means correct in the sense implied in the working of the examples given above; that is, that the figures in the approximation express a result that is closer to the true value than can be expressed by the same number of any other figures.

The following is the rule observed when figures are omitted:

When the figure omitted is 5 *or over* 5, *add* 1 *to the last figure retained; and, when the figure omitted is under* 5, *make no change in the last figure retained.*

If, in accordance with this rule, 1 be added for a rejected figure, the maximum error is ·05 for a digit rejected in the second decimal place, ·005 for a digit rejected in the third decimal place, ·0005 for a digit rejected in the fourth decimal place; and so on, the error in each case being one of excess. Obviously it is greatest when the rejected digit is 5 and least when the rejected digit is 9.

On the other hand, if, in accordance with the rule, figures under 5 are rejected, the maximum error is ·04 for a digit rejected in the second decimal place, ·004 for a digit rejected in the third decimal place, ·0004 for a digit rejected in the fourth decimal place; and so on, the error in each case being one of defect. Obviously it is greatest when the rejected digit is 4 and least when the rejected digit is 1. Of course the magnitude of the error depends partly on the values of the digits following the rejected figure.

The digits 1 to 9 are called *significant figures* wherever they are placed, and 0 is a *significant figure* when it is placed between any of the digits 1 to 9. Thus, in the numbers 7025 and 7·025 the 0 is significant, the significant figures in each case being 7025.

$$76300 = 763 \times 10^2,$$
$$\cdot 00763 = 763 \times 10^{-5},$$

where the 0's indicate powers of 10 or the order of the number 763 in each case. They are therefore not significant figures.

It is necessary to discriminate between "correct to x decimal places" and "correct to x significant figures."

Correct to 3 decimal places, 46·7857 = 46·786;
correct to 3 significant figures, 46·7857 = 46·8;
correct to 2 decimal places, ·0467 = ·05;
correct to 2 significant figures, ·0467 = ·047.

$$\text{Similarly, } 387653 = \begin{cases} 390000, \text{ correct to 2 significant figures,} \\ 388000, \text{ correct to 3 significant figures,} \\ 387700, \text{ correct to 4 significant figures,} \\ 387650, \text{ correct to 5 significant figures.} \end{cases}$$

EXERCISES. (*Oral.*)

(*a*) Give from 1 decimal place to 5 decimal places the nearest approximations to each of the following:

1. 2·768139. **2.** ·368495. **3.** 7·254784.

(*b*) Read off the following, correct to 3 significant figures:

1. 2·736. **2.** ·07648. **3.** 5·006. **4.** 58374.

As we have seen, in a number that is "correct" to 3 decimal places, the error cannot be greater than ·0005, an error so small that for most practical purposes it may be disregarded. In the smaller units with which we have to deal, the difficulties of exact measurement are so great that only skilled workers can, with the best instruments, attain to four—or five—place accuracy; and therefore decimal expressions that extend beyond four or five places represent fancied rather than actual accuracy; but, whether the accuracy of the figures of a decimal be fancied or real, it is often the case that more decimal places are given than are necessary for purposes of computation to degrees of accuracy that are of practical value. What we have to do in decimal work is to make sure that the magnitude of the error in the final result is very small in comparison with the quantity the measurements of which are attained or given. For example, ·005 of £1 is less than the smallest coin of the realm, while ·005 of £10,000 is £50. In a final result the first may generally be disregarded, but not the second.

It now remains to exemplify the devices adopted to find results sufficiently accurate for practical purposes without incurring the labour of finding unnecessary figures.

Approximate sum or difference.

Ex. 1. *Find, correct to* 2 *decimal places, the sum of* 2·76745, 17·68, ·00763, 8·5916, 4·03459.

(*a*)

$$\begin{array}{r|l} 2{\cdot}76 & 745 \\ 17{\cdot}68 & \\ {\cdot}00 & 763 \\ 8{\cdot}59 & 16 \\ 4{\cdot}03 & 459 \\ \hline 33{\cdot}08 & 1 \end{array}$$

= 33·08, correct to 2 places.

(*b*)

$$\begin{array}{r} 2{\cdot}77 \\ 17{\cdot}68 \\ {\cdot}01 \\ 8{\cdot}59 \\ 4{\cdot}03 \\ \hline 33{\cdot}08 \end{array}$$

In (*a*), instead of approximating each separate number as in (*b*), we have operated on *four places* of decimals, *two places* more than the limit imposed by the conditions of the question. This is the method the student should generally adopt. The rule is:

Work to two more places than are required, and strike out of the answer the last two places, increasing by 1 *the last figure retained if the next figure after it is* 5 *or more than* 5.

Ex. 2. *Find, correct to 3 decimal places, the difference between* 25·623457 *and* 19·378695.

$$\begin{array}{r|l} 25{\cdot}623 & 457 \\ 19{\cdot}378 & 695 \\ \hline 6{\cdot}244 & 7 \end{array}$$

$= 6{\cdot}245$, correct to 3 places.

EXAMPLES. XLIII.

Find, correct to 1, 2, 3, and 4 places, the value of the sum of:

1. 8·756946, ·069878, 17·843245, 26·695769.

2. 6·08749, 8·965, 28·0958, ·95, ·069.

Find, correct to 4 places, the difference between:

3. 7·065874 and 3·27499.

4. 4·568536 and 2·959697.

5. 1·007445 and ·988774.

6. 10·356 and ·978326.

It is sometimes necessary to find the decimal value of the sum of a series of vulgar fractions. In such cases much trouble is saved by making each preceding result subservient to the next following. The following example will serve to illustrate what we mean.

Ex. *Find, correct to 6 places, the decimal equivalent of*

$$\frac{1}{2\,.\,5}+\frac{1}{2\,.\,5\,.\,8}+\frac{1}{2\,.\,5\,.\,8\,.\,11}+\text{etc.}$$

Observe here (1) that in each succeeding fraction an additional factor is introduced into the denominator, and (2) that each of these factors is greater than the last by the constant difference. (3) The additional factor may, of course, be any number or power of any number.

$$\text{Since } \frac{1}{2\,.\,5\,.\,8}=\frac{1}{8}\text{ of }\frac{1}{2\,.\,5},$$

$$\frac{1}{2\,.\,5\,.\,8\,.\,11}=\frac{1}{11}\text{ of }\frac{1}{2\,.\,5\,.\,8},\text{ and so on,}$$

it follows that $\frac{1}{8}$ of the decimal representing $\frac{1}{2.5}$

$= \text{the decimal equivalent of } \frac{1}{2.5.8}$;

also $\frac{1}{11}$ of the decimal representing $\frac{1}{2.5.8}$

$= \text{the decimal equivalent of } \frac{1}{2.5.8.11}$, and so on.

$$\begin{array}{llr}
\text{Now,} & \frac{1}{2.5} & = \cdot 10000000 \\
\therefore & \frac{1}{2.5.8} & \frac{\cdot 10000000}{8} = \cdot 01250000 \\
 & \frac{1}{2.5.8.11} & = \frac{\cdot 01250000}{11} = \cdot 00113636 \\
 & \frac{1}{2.5.8.11.14} & = \frac{\cdot 00113636}{14} = \cdot 00008117 \\
 & \frac{1}{2.5.8.11.14.17} & = \frac{\cdot 00008117}{17} = \cdot 00000477 \\
 & \frac{1}{2.5.8.11.14.17.20} & = \frac{\cdot 00000477}{20} = \underline{\cdot 00000023} \\
 & & \therefore \text{ the sum} = \cdot 11372253
\end{array}$$

$\therefore$ sum accurate to 6 places $= \cdot 113723$.

Obviously the last step in the series might have been omitted, since it does not affect the sixth decimal place of the sum of the series.

EXAMPLES. XLIV.

1. Reduce to a decimal, accurate to seven places,

$$\frac{1}{1.3} + \frac{1}{1.3.5} + \frac{1}{1.3.5.7} + \text{etc.}$$

2. Find to seven decimal places the value of the series

$$\frac{1}{1.2} - \frac{1}{1.2.3} + \frac{1}{1.2.3.4} - \frac{1}{1.2.3.4.5} + \text{etc.}$$

3. Find to six places of decimals the value of

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \text{etc.}$$

4. Find to four decimal places the value of

$$2\left(\frac{1}{7} + \frac{1}{3} \times \frac{1}{7^3} + \frac{1}{5} \times \frac{1}{7^5} + \frac{1}{7} \times \frac{1}{7^7} + \text{etc.}\right).$$

Degree of accuracy in Addition and Subtraction.

If n quantities are given as correct to two decimal places, each of them may be too small or too large by an amount not greater than ·005; it follows therefore that, if the quantities be added, the maximum error of excess will be not greater than $\cdot 005 \times n$, and the maximum error of defect not greater than $\cdot 005 \times n$; that is, the limits of error will be $\pm \cdot 005 \times n$.

Ex. 1. *In each of the numbers* 7·83, 6·95, *and* 4·68 *it is known that there is a possible error of* ·005: *what are the limits of error in the sum, and between what limits must the true sum lie?*

Sum.

$$\begin{array}{r} 7{\cdot}83 \\ 6{\cdot}95 \\ 4{\cdot}68 \\ \hline 19{\cdot}46 \end{array} \qquad \begin{aligned} \text{Limits of error} &= \pm {\cdot}005 \times n \\ &= \pm {\cdot}005 \times 3 \\ &= \pm {\cdot}015. \end{aligned}$$

Hence the two limits between which the true sum lies are 19·46 + ·015 and 19·46 − ·015, that is, 19·475 and 19·445.

Hence the sum 19·46 can be relied upon only to the first decimal place.

Ex. 2. *Two lengths are given as* 12·345 *metres and* 7·437 *metres correct to* 3 *places: to what degree of accuracy can their difference be computed?*

Difference.

$$\begin{array}{r} 12{\cdot}345 \\ 7{\cdot}437 \\ \hline 4{\cdot}908 \end{array} \qquad \begin{aligned} \text{Limits of error} &= \pm {\cdot}0005 \times 2 \\ &= \pm {\cdot}001. \end{aligned}$$

Hence the difference can be relied upon only to the second decimal place.

EXAMPLES. XLV.

To what degree of accuracy can the following be computed? State the *limits* of error in each case:

1. 27·35 + 5·91 + 6·78 + ·75 (correct to 2 places).
2. 7·68 + 4·87 + 5·32 + ·86 + ·08 (correct to 2 places).
3. 4·581 + 6·753 + 8·964 + 2·436 (correct to 3 places).
4. 17·321 metres − 12·382 metres (correct to mm.).
5. 14·275 grams − 9·396 grams (correct to mg.).
6. 7·36 g. − 3·97 g. (correct to cg.).

Approximate Products.

Ex. 1. *Find* (a) *to the nearest integer,* (b) *to* 1 *decimal place, and* (c) *to* 3 *decimal places, the value of* 18·76586 × 7.

$$\begin{array}{ccc}
(a) & (b) & (c) \\
\begin{array}{r|l} 18 & \cdot 76586 \\ 7 & \\ \hline 131 & \cdot 3 \end{array} &
\begin{array}{r|l} 18\cdot 7 & 6586 \\ 7 & \\ \hline 131\cdot 3 & 6 \end{array} &
\begin{array}{r|l} 18\cdot 765 & 86 \\ 7 & \\ \hline 131\cdot 361 & 0 \end{array} \\
\textit{Answer } 131. & \textit{Answer } 131\cdot 4. & \textit{Answer } 131\cdot 361.
\end{array}$$

To be as accurate as possible, we must always allow for the *products* of the *rejected digits.* In (*a*) the multiplication begins with 7 × 6, two places to the right of the last figure to be retained : 7 × 6 = 42, carry 4 ; 7 × 7 = 49, which with 4 makes 53 ; carry 5 ; and so on. In (*b*) the multiplication begins with 7 × 5, which is 35 ; carry 4 ; and so on. In all such cases, carry 1 for every 10 in the product, that is, carry the nearest ten. Thus, in 21, 22, 23, 24, 2 is the nearest ten, while in 26, 27, 28, 29, 3 is the nearest ten. For 5, 15, 25, 35, etc., carry 1, 2, 3, 4, etc., respectively. In other words : *When the units' digit is below* 5, *carry the lower ten* ; *and, when it is over* 4, *carry the higher ten.*

The following results should be carefully noted :

$$a \times b = ab,$$

$$a \times 10 \times \frac{b}{10} = ab,$$

$$\frac{a}{10} \times b \times 10 = ab,$$

$$\frac{a}{10^x} \times b \times 10^x = ab.$$

That is, *the product of two factors is not altered when we multiply one and divide the other by the same number.*

Hence we may change the multiplier and multiplicand to any convenient form by multiplying the one and dividing the other by the same power of 10. As a rule, the most convenient form for the multiplier is what is called the *standard form,* that is, the form in which the highest significant figure occupies the units' place, as 5·07.

586·4 expressed in standard form is $5\cdot 864 \times 10^2$;

·00538 expressed in standard form is $5\cdot 38 \times 10^{-3}$.

Ex. 2. *Find, correct to* 3 *decimal places, the product of* ·356174 *and* 49·5.

$$·356174 \times 49·5 = 3·56174 \times 4·95.$$

To multiply a number by 4·95 is to multiply it (1) by 4, (2) by ·9, and (3) by ·05, and then sum the products.

An answer correct to thousandths is correct to 3 places; but, to make reasonably certain of accuracy in the third place, we must take account of one or two more places.

Now, 3·5617 . × 4 yields ten-thousandths,
3·561 . . × ·9 yields ten-thousandths,
3·56 . . . × ·05 yields ten-thousandths.

That is, we shall get ten-thousandths in each partial product even if we reject in the multiplicand one digit for the units' multiplier, two digits for the tenths' multiplier, and three digits for the hundredths' multiplier. Allowance must, however, be made for the products of the rejected digits. How this is done is shown below:

$$\begin{array}{r l}
3·56174 & \\
\underline{4·95} & \\
14·2470 & = \left\{\begin{array}{l}3·5617 \times 4, \text{ with allowance for the product of the rejected} \\ \quad 4. \text{ Say four 4's, 16, carry 2.}\end{array}\right. \\
3·2055 & = \left\{\begin{array}{l}3·561 \times ·9, \text{ with allowance for the product of the rejected} \\ \quad 7. \text{ Say nine 7's, 63, carry 6.}\end{array}\right. \\
\underline{·1781} & = \left\{\begin{array}{l}3·56 \times ·05, \text{ with allowance for the product of the rejected} \\ \quad 1. \text{ Say five 1's, 5, carry 1.}\end{array}\right. \\
17·6306 &
\end{array}$$

= 17·631, correct to three decimal places.

Or thus, the multiplier being inverted:

$$\begin{array}{r}
3·56174 \\
\underline{594} \\
14·2470 \\
3·2055 \\
\underline{·1781} \\
17·6306
\end{array}$$

The only difference between this method and the first lies in the *inversion* of the multiplier, which renders the work much less trying to the eye.

When it is adopted, *the units' digit of the multiplier is placed under the last decimal digit retained in the multiplicand, and each partial product is so placed that its right-hand digit falls under the units' digit of the multiplier.*

The student is recommended to express the multiplier in "standard form."

Ex. 3. *Multiply* 37·965 *by* ·00244, *so that the result may be correct to within* ·00005.

$$·00244 = 2·44 \times 10^{-3};$$

$$\therefore\ 37·965 \times ·00244 = ·037965 \times 2·44.$$

$$\begin{array}{r}
·037965 \\
\underline{2·44} \\
·07593 \\
1518 \\
\underline{152} \\
·09263
\end{array}
\qquad\Bigg|\qquad
\begin{array}{r}
·037965 \\
\underline{442} \\
·07593 \\
1518 \\
\underline{152} \\
·09263
\end{array}$$

Answer, ·0926.

Ex. 4. *Find to 3 significant figures the product of* 7586975 *and* 97687.

7586975	7586975
97687	78679
68282	68282
5310	5310
455	455
60	60
5	5
74112	74112

∴ the three significant figures are 741.

We have now to determine what they signify.

The 75 of the multiplicand is 75×10^5,

the 9 of the multiplier is 9×10^4,

the 741 of the product is therefore 741×10^9,

that is, 741,000,000,000.

EXAMPLES. XLVI.

1. Multiply 17·05 by ·562903, retaining 5 decimal places.
2. Multiply 416·20159 by ·008907, retaining 5 decimal places.
3. Multiply 10·89 by $·486\dot{4}\dot{6}$, retaining 4 decimal places.
4. Multiply 1·00123 by 1·165, retaining 5 decimal places.
5. Multiply 3·00596891 by ·46973, retaining 6 decimal places.
6. Find the value of $7·986 \times 2·09 \times 4·723$, retaining 3 decimal places.

Degree of accuracy in products.

Ex. *A rectangular surface is measured and found to be* 5·71 *metres long and* 3·34 *metres wide, correct to centimetres: find the limits of error in the area, and to what degree of accuracy the area can be computed.*

Let l represent the length, b the breadth, and x the maximum possible error in length and breadth; then the true length cannot exceed $(l+x)$ metres, and the true breadth cannot exceed $(b+x)$ metres.

Hence the true area cannot exceed $(l+x)(b+x)$ metres,

that is, $lb+bx+lx+x^2$,

that is, $lb+(b+l)x+x^2$.

$x^2=(·005)^2$, a result relatively so small that it may be neglected.

Also the true area cannot be less than $(l-x)(b-x)$ metres,

that is, $lb-bx-lx+x^2$,

that is, $lb-(b+l)x+x^2$.

Hence, neglecting x^2, we have

$$\begin{aligned}\text{limits of errors} &= \pm(b+l)x \\ &= \pm(3{\cdot}34+5{\cdot}71)\times{\cdot}005 \\ &= \pm{\cdot}04525 \text{ of a sq. metre.}\end{aligned}$$

According to the measured dimensions,

$$\begin{aligned}\text{area of surface} &= (5{\cdot}71\times 3{\cdot}34) \text{ sq. metres} \\ &= 19{\cdot}0714 \text{ sq. metres.}\end{aligned}$$

$\therefore$ the area lies between $19{\cdot}0714+{\cdot}04525$,
and $19{\cdot}0714-{\cdot}04525$,
that is, between $19{\cdot}11665$ and $19{\cdot}02615$.

Hence accuracy can be relied upon only to metres, that is, to the units' place.

Generally, if the maximum possible errors in the factors are equal, the limits of error may be expressed by $\pm\ (a+b)x$, where a and b are the factors, and x the value of the maximum error in each of the factors.

Suppose we have factors correct to different numbers of places, as a involving a possible maximum error of x, and b a possible maximum error of y; then

the product cannot exceed $(a+x)(b+y)$,
that is, $ab+bx+ay+xy$;
and it cannot be less than $(a-x)(b-y)$,
that is, $ab-bx-ay+xy$.

xy is relatively so small that it may be neglected;
hence the limits of error are $\pm(ay+bx)$.

EXAMPLES. XLVII.

1. The sides of a rectangle are known to be 4·75 metres and 3·67 metres, correct to centimetres: to how many correct figures can the area of the rectangle be calculated, and what are the limits of error?

2. The sides of a rectangle are given as 2·563 metres and 4·356 metres, correct to mm.: to how many correct figures can the area of the rectangle be calculated, and what are the limits of error?

3. The sides of a rectangular piece of paper are given as 5·67 inches long and 4·563 inches wide, correct to as many figures as are given: to how many correct figures can the area be calculated, and what are the limits of error?

4. 237·61 and 18·5 are given as correct to as many figures as are given: find the limits between which their nearest approximate product lies.

5. If a quantity that is correct to three decimal places is multiplied by 28, what are the limits of error in the product?

Approximate Quotients.

Ex. 1. *Find the quotient of* 13·25743 *divided by* 7, *correct* (a) *to tenths,* (b) *to hundredths, and* (c) *to thousandths.*

(*a*)	(*b*)	(*c*)
1·89...	1·893...	1·8939...
7) 13·25743	7) 13·25743	7) 13·25743
Answer, 1·9, correct within ·05.	*Answer*, 1·89, correct within ·005.	*Answer*, 1·894, correct within ·0005.

Ex. 2. *Find the quotient of* 932·58 *divided by* 24·8153, *correct to within* ·0005.

In examples of this kind we can, after a certain number of quotient figures have been obtained in the usual way, obtain the rest by cutting off digits from the right of the divisor instead of affixing 0's to the right of the dividend. *We begin to cut off the digits when the digits in the divisor are* **two more** *than the number of digits still to be obtained in the quotient.*

A glance at the example shows that there will be two digits in the integral part of the quotient, and that therefore the entire quotient will contain *five* figures; but, as we cannot be sure of the fifth figure without knowing the sixth, we must find the quotient to six figures. We have therefore a six-figure divisor and a six-figure quotient; hence we begin to reject figures in the divisor after two figures have been obtained in the quotient. The working of the example will make the method clear.

Expressed in terms of the same order, the numbers are 932·5800 and 24·8153.

$$\begin{array}{r} 37{\cdot}5808 \\ 24,{\cdot}8,1,5,3\;\overline{)\,932{\cdot}5800} \\ 188\,1210 \\ 14\,4139 \\ 2\,0062 \\ 210 \\ 12 \end{array}$$

Or thus:

$$\begin{array}{r} 37{\cdot}5808 \\ 24,{\cdot}8,1,5,3\;\overline{)\,932{\cdot}5800} \\ \underline{744\,459} \\ 188\,1210 \\ \underline{173\,7071} \\ 14\,4139 \\ \underline{12\,4077} \\ 2\,0062 \\ \underline{1\,9852} \\ 210 \\ \underline{198} \\ 12 \end{array}$$

Notes on the working: *We must use rejected figures to find out what must be carried.*

Contraction begins at the tenths' digit of the quotient.

1st step: Reject 3 of divisor, allowing 2 for 5 times the rejected figure $(3 \times 5 = 15)$; five 5's, 25 and 2, 27; and so on.

2nd step: Reject 5 of divisor, allowing 4 for 8 times the rejected figure ($5 \times 8 = 40$); eight 1's, 8 and 4, 12; and so on.

3rd step: Reject 1 of divisor; divisor not contained in dividend; put 0 in quotient.

4th step: Reject 8 of divisor, allowing 6 for 8 times the rejected figure ($8 \times 8 = 64$); eight 4's, 32 and 6, 38; and so on.

EXAMPLES. XLVIII.

1. Divide 875·1 by ·0103, retaining integers only.

2. Divide 1301·11 by 153·761, retaining 3 places of decimals.

3. Divide 4·02354691 by 72·4673, retaining 6 places of decimals.

4. Divide ·0349564 by ·650357, retaining 6 places of decimals.

5. Divide 5·74 by 583·26754, retaining 6 places of decimals.

6. Divide 64·20153 by ·9876, retaining 5 places of decimals.

Degree of accuracy in division.

Ex. *The area of a rectangular flower plot is* 110·71 *square metres, correct to square centimetres, and its width is* 7·25 *metres, correct to centimetres: find between what limits the length must lie.*

$$\text{Greatest possible length} = \frac{110{\cdot}71 + {\cdot}005}{7{\cdot}25 - {\cdot}005}\text{ m.}$$

$$= 15{\cdot}28\ldots\text{ m.}$$

$$\text{Least possible length} = \frac{110{\cdot}71 - {\cdot}005}{7{\cdot}25 + {\cdot}005}\text{ m.}$$

$$= 15{\cdot}26\text{ m. nearly.}$$

Therefore the length lies between the limits 15·28... m. and 15·26 m. (nearly), and its accuracy can be depended on only to decimetres.

The general case.

Let a represent the dividend and b the divisor to the same degree of accuracy, and let $\pm x$ represent the limits of error; then

$$\text{the greatest possible quotient} = \frac{a+x}{b-x},$$

$$\text{and the least possible quotient} = \frac{a-x}{b+x};$$

from which it is easy to find the limits of error in the quotient $\frac{a}{b}$:

for possible errors in first case $= \dfrac{a+x}{b-x} - \dfrac{a}{b}$

$$= \frac{b(a+x) - a(b-x)}{b(b-x)}$$

$$= \frac{ax+bx}{b(b-x)}$$

$$= \frac{(a+b)x}{b^2-bx}.$$

Being relatively very small, bx may be neglected.

Hence possible error $= \dfrac{(a+b)x}{b^2}$ approx.

Possible error in second case $= \dfrac{a}{b} - \dfrac{a-x}{b+x}$

$$= \frac{a(b+x) - b(a-x)}{b(b+x)}$$

$$= \frac{ax+bx}{b^2+bx}.$$

Neglecting bx as before, we have

possible error $= \dfrac{(a+b)x}{b^2}$ approx.

∴ the limits of error are $\pm\dfrac{(a+b)x}{b^2}$ approx.

Let us apply this result to the example worked above, where $a = 110{\cdot}71$, $b = 7{\cdot}25$, and $\pm x = {\cdot}005$,

$\dfrac{a}{b} = \dfrac{110{\cdot}71}{7{\cdot}25}$ m. $= 15{\cdot}27$ m., correct to 2 places.

The limits of error are $\pm\dfrac{(a+b)x}{b^2}$ approx.;

that is, $\pm\dfrac{(110{\cdot}71 + 7{\cdot}25) \times {\cdot}005}{(7{\cdot}25)^2}$;

that is, $\pm\dfrac{{\cdot}5898}{52{\cdot}5625}$ or $\pm {\cdot}01$ nearly.

Therefore the length lies between $(15{\cdot}27 + {\cdot}01)$ m. and $(15{\cdot}27 - {\cdot}01)$ m.; that is, between 15·28 and 15·26.

Suppose the divisor b and the dividend a to be correct to different numbers of places, and let $\pm x$ represent the limits of error in the dividend, and ± y the limits of error in the divisor; then

$$\text{the greatest possible quotient} = \frac{a+x}{b-y},$$

$$\text{and the least possible quotient} = \frac{a-x}{b+y}.$$

$$\begin{aligned}\text{Possible error in first} &= \frac{a+x}{b-y} - \frac{a}{b}\\ &= \frac{b(a+x)-a(b-y)}{b(b-y)}\\ &= \frac{ay+bx}{b^2}\ \text{approx.};\end{aligned}$$

$$\begin{aligned}\text{possible error in second} &= \frac{a}{b} - \frac{a-x}{b+y}\\ &= \frac{a(b+y)-b(a-x)}{b(b+y)}\\ &= \frac{ay+bx}{b^2}\ \text{approx.};\end{aligned}$$

$$\therefore \text{ the limits of error are } \pm\frac{ay+bx}{b^2}\ \text{approx.}$$

EXAMPLES. XLIX.

1. Find how many figures are trustworthy in the quotients of the following: (*a*) 126·74 ± ÷ 57·35 ±. (*b*) 17·384 ± ÷ 2·25 ±.

2. The area of a field is 2756 square metres to the nearest square metre, and the length of one side is found by measurement to be 37·6 metres to the nearest decimetre: find to 3 figures the limits between which the other side must lie.

3. The cost of a block of land of 24·2 acres (correct to tenths) is £304 (correct to the nearest £): find to farthings the limits between which the price per acre lies.

4. The number of posts in a stack is guaranteed to be 5300 to the nearest hundred, and the price is guaranteed to be within £5 of £380: find the price-per-post limits to two decimal places of a penny.

5. In the following division the dividend is known to be correct to three places and the divisor to four places of decimals: to how many decimal places can the quotient be relied on?

6. If 1·076, correct to the third decimal place, be divided by ·08, correct to the second decimal place, between what limits will the quotient lie?

7. Without the cover, the thickness of a book is 0·73 ± inch, and the thickness of a leaf is 0·006 ± inch: find the maximum and the minimum number of leaves it can contain.

CHAPTER X.

THE H.C.F. AND THE L.C.M. OF FRACTIONS.

THE H.C.F. of two or more fractions is the largest fraction that will divide exactly into each of the given fractions, from which it follows that the H.C.F. of two or more fractions is a fraction the numerator of which will divide exactly into the numerators of the given fractions, and the denominator of which will contain exactly the denominators of the given fractions.

Hence the numerator of the H.C.F. is the H.C.F. of the numerators and the denominator of the H.C.F. is the L.C.M. of the denominators of the given fractions.

That is, the H.C.F. of the fractions $= \dfrac{\text{H.C.F. of numerators}}{\text{L.C.M. of denominators}}$.

Ex. 1. *Find the* H.C.F. *of* $6\frac{3}{4}$, $1\frac{7}{8}$, *and* $13\frac{5}{7}$.

The fractions $= \frac{27}{4}, \frac{15}{8}, \frac{96}{7}$.

The H.C.F. of the numerators is 3,

the L.C.M. of the denominators is 56;

$\therefore$ the H.C.F. of the given fractions is $\frac{3}{56}$.

Note. Before proceeding to find the H.C.F. or the L.C.M. of fractions, we must see that the fractions are expressed in their lowest terms.

Just as the L.C.M. of two or more whole numbers is the smallest number that will contain each of them an exact number of times, so the L.C.M. of two or more fractions is the smallest fraction that will contain each of the fractions an exact number of times.

Hence the L.C.M. of two or more fractions is a fraction the numerator of which is the L.C.M. of the numerators of the given fractions and the denominator of which is the H.C.F. of their denominators.

That is, the L.C.M. of the fractions $= \dfrac{\text{L.C.M. of numerators}}{\text{H.C.F. of denominators}}$.

Ex. 2. *Find the* L.C.M. *of* $1\frac{5}{28}$, $2\frac{2}{21}$, $3\frac{13}{14}$, *and* $3\frac{1}{7}$.

The fractions $=\frac{33}{28}, \frac{44}{21}, \frac{55}{14}, \frac{22}{7}$.

The L.C.M. of the numerators is 660,

the H.C.F. of the denominators is 7;

∴ the L.C.M. of the fractions is $\frac{660}{7}$, or $94\frac{2}{7}$.

Ex. 3. *Find the* H.C.F *and the* L.C.M. *of* ·06, ·008, *and* ·16.

·06, ·008, ·16 $=\frac{3}{50}, \frac{1}{125}, \frac{4}{25}$.

The H.C.F. of the numerators is 1,

the L.C.M. of the denominators is 250;

∴ the H.C.F. of the decimals is $\frac{1}{250}$, or ·004.

Or thus:

To get rid of the decimals, multiply by 1000; then ·06, ·008, ·16 become 60, 8, 160, the H.C.F. of which is obviously 4; but, since the given decimals were multiplied by 1000, the factor 4 must be 1000 times the required factor; therefore the H.C.F. is $\frac{4}{1000}$, or $\frac{1}{250}$, or ·004.

Again:

The L.C.M. of the numerators of the fractions $\frac{3}{50}$, $\frac{1}{125}$, and $\frac{4}{25}$ is 12;

the H.C.F. of the denominator is 25;

∴ the L.C.M. of the fractions is $\frac{12}{25}$, or ·48.

Or, multiplying the given decimals by 1000 we have, as before, 60, 8, 160, the L.C.M. of which is obviously 480; but, since the given decimals were multiplied by 1000, the multiple 480 must be 1000 times the required multiple; therefore the L.C.M. is $\frac{480}{1000}$, or $\frac{12}{25}$, or ·48.

Ex. 4. *One man's step is $\frac{3}{4}$ yd., another's $\frac{7}{8}$ yd., and another's $\frac{9}{16}$ yd.: if they start walking at the same instant, how far will they walk before they are in step again?*

The L.C.M. of the fractions $\frac{3}{4}, \frac{7}{8}, \frac{9}{16}$ is the number of yds. they must walk to be in step again.

The L.C.M. of the numerators is 63,

the H.C.F. of the denominators is 4;

∴ the L.C.M. is $\frac{63}{4}$, or $15\frac{3}{4}$ yards.

Or, multiplying the fractions by 16, the smallest number that will make them all whole numbers, we have 12, 14, 9, the L.C.M. of which is 252; but, since 12, 14, 9 are respectively 16 times the fractions they represent, 252 is 16 times the L.C.M. of those fractions;

∴ L.C.M. $=\frac{252}{16}=15\frac{3}{4}$ yards.

EXAMPLES. L.

1. Find the L.C.M. of 6·4, 7·2, and 4·8.

2. Two persons take steps of 2·21 and 1·95 feet: if they start in step, how far will they have walked before they are in step again?

3. One pendulum oscillates in $\frac{57}{58}$ of a second, and another in $\frac{76}{87}$ of a second: how often will they be together in 1 hour?

4. Five bells begin to toll together, and they toll respectively at intervals of $7\frac{1}{2}$, $7\frac{1}{3}$, $6\frac{3}{4}$, $6\frac{2}{3}$, and $5\frac{1}{2}$ seconds: in what time will they all toll together again?

5. Three girls begin skipping at the same time: they jump at intervals of ·6, ·75, ·8 seconds respectively: in what time will they again jump together?

6. Express the H.C.F. of $\frac{3}{7}$, $\frac{5}{14}$, $\frac{4}{21}$, $\frac{3}{35}$ as a fraction of the L.C.M. of the same fractions.

7. *A* and *B* start in step and walk side by side, *A*'s step being $2\frac{13}{22}$ feet and *B*'s $2\frac{7}{44}$: how far will they have gone when they are in step again?

8. What is the least length that is a multiple of 11 feet, 14 feet 8 inches, and 16 feet 6 inches?

9. Find the number that exceeds its sixth part as much as 26 exceeds the fourth part of it.

10. If to the sum of $\frac{3}{14}$ and $\frac{5}{8}$ of a number 34 be added, the sum is 128: find the number.

11. If from the sum of $\frac{5}{6}$ and $\frac{7}{10}$ of a number 41 be taken, the remainder is 97: find the number.

12. If $\frac{7}{18}$ of my money be multiplied by 2, and £11 be taken from the product, the remainder is £45: how much money have I?

13. After spending $\frac{3}{5}$ of his money, a man pays a bill of £40 and one of £56, and then has £144 left: how much had he at first?

14. If $a=4$, $b=3$, $c=2$, and $y=5$, find the value of x in the equation:

$$ax+xy=ab+bc.$$

15. A man walks m miles in n hours: how many miles does he walk in 1 hour?

Also how many minutes does he take to walk 1 mile, and how many hours to walk x miles?

CHAPTER XI.

PERIODIC OR RECURRING DECIMALS.

We have seen that, if the only prime factors of the denominator of a vulgar fraction expressed in its lowest terms are 2 or 5, or 2 and 5, a decimal equivalent in value to the fraction can be found by dividing the denominator into the numerator multiplied by some power of 10.

For example $\frac{3}{8}=\frac{3{\cdot}000}{8}={\cdot}375$

$$=\frac{3}{10}+\frac{7}{100}+\frac{5}{1000}.$$

At the last step the division ends without remainder; the decimal is therefore called a *terminating* or *finite* decimal.

Try the fraction $\frac{7}{12}$, which is equal to $\frac{7}{2^2\times 3}$.

$$\frac{7}{12}=\frac{7{\cdot}000000}{12}={\cdot}583333\ldots,$$

the remainder 4 recurring without end.

It is obvious that the factor 3 produces this result, for $\frac{4}{12}=\frac{1}{3}$.

Observe that the number of 2's in the denominator is also the number of non-recurring digits in the decimal.

As, however far we carry the division, it does not end; the quotient is called a *non-terminating* decimal.

Our decimal is accurate to the fifth place. The following analysis shows the value of each digit and of the remainder;

$${\cdot}583333\ldots=\frac{5}{10}+\frac{8}{10^2}+\frac{3}{10^3}+\frac{3}{10^4}+\frac{3}{10^5}+\frac{3}{10^6}+\frac{1}{3}\text{ of }\frac{1}{10^6}.$$

Now, $\frac{1}{3}$ of $\frac{1}{10^6}=\frac{1}{3000000}$,

that is, the decimal expression ·583333 expresses the value of a unit to within ·0000003 of the unit.

It is clear that the farther we carry the division, the nearer we approach to accuracy; but that, however far we carry it, the sum of the series of fractions $\frac{5}{10}+\frac{8}{10^2}+\frac{3}{10^3}+\dots$ is never equal to $\frac{7}{12}$.

Regarded from this point of view, $\frac{7}{12}$ is the sum of an infinite series of fractions $\frac{5}{10}$, $\frac{8}{100}$, $\frac{3}{1000}$, $\frac{3}{10000}$, etc., the sum of the addends of which approximates more and more closely to $\frac{7}{12}$ the more addends we take, but never equals it nor goes beyond it. The fraction $\frac{7}{12}$ is therefore called the *limit* of the infinite series $\frac{5}{10}+\frac{8}{10^2}+\frac{3}{10^3}+\frac{3}{10^4}+\dots$.

If we stop at the third member of the series, the defect is $\frac{1}{3}$ of $\frac{1}{10^3}$, that is, $\frac{1}{3000}$ or ·0003 of the unit; if we stop at the fourth member of the series, the defect is $\frac{1}{3}$ of $\frac{1}{10^4}$, that is, $\frac{1}{30000}$ or ·00003 of the unit; if we stop at the fifth member of the series, the defect is $\frac{1}{3}$ of $\frac{1}{10^5}$, that is, $\frac{1}{300000}$ or ·000003 of the unit, the defect becoming less and less as we pass to the right. It is therefore obvious that we can make the sum of the series differ from $\frac{7}{12}$ by as small a defect as we please.

Decimal expressions of the kind we are now considering do not occur in actual measurements; they occur only in connexion with the reduction of vulgar fractions to decimal equivalents.

The notation of the decimal ·583333... is $\cdot 58\dot{3}$, the dot over the 3 signifying that the 3 repeats without end.

Another example: $\frac{6}{35}$, which is equal to $\frac{6}{5\times 7}$.

$$\frac{6}{35}=\cdot 1714285714285\dots,$$

where, after the first quotient figure, the same sequence of remainders and therefore the same period of quotient figures recur without end.

Observe that there is one 5 in the denominator of the fraction $\frac{6}{5\times 7}$ and one non-recurring figure in the decimal. The recurring period is obviously due to the presence of 7 in the denominator.

The notation of the decimal is $\cdot 1\dot{7}1428\dot{5}$, the dot over the 7 and the 5 signifying that the period 714285 repeats without end.

The recurring period begins when the remainder is $\frac{25}{35}$, that is, $\frac{5}{7}$, which is therefore called the *generating fraction* of the recurring period.

If the denominator of the generating fraction expressed in its lowest terms is represented by n, then the recurring period cannot contain more than $(n-1)$ digits, but it may contain fewer. The law may be stated thus:

If $\frac{m}{n}$ *be a proper fraction in its lowest terms and* n *be a prime number other than* 2 *or* 5, *then* $\frac{m}{n}$ *generates a recurring period of* $(n-1)$ *digits or a period of some factor of* $(n-1)$ *digits.*

In the last example the period contains 6 digits, that is, $(7-1)$ digits.

Let it be required to find the decimal generated by $\frac{3}{11}$.

The period cannot contain more than $(11-1)$ digits, but it may contain fewer, namely, a number that is some factor of $(11-1)$.

$11-1=10$, the factors of which are 2 and 5 or 1 and 10.

$$\frac{3}{11}=\frac{3\cdot 0000\ldots}{11}=\cdot 2727\ldots$$
$$=\cdot\dot{2}\dot{7}.$$

It is an easy inference from the foregoing examples that the number of digits in the period depends only on the denominator of the generating fraction.

Decimals in which digits recur in the same sequence are called *indeterminate*, *repeating*, *circulating*, or *recurring decimals.*

In the last example worked above the recurring period begins immediately after the decimal point; it is therefore called a *pure recurring decimal.*

In the other examples there are immediately after the decimal point one or more digits that do not form part of the recurring period. Such examples are called *mixed recurring decimals.*

It is necessary to discriminate between the generating fraction of a decimal and the generating fraction of a recurring period.

In the example $\frac{7}{12} = \cdot 58\dot{3}$, the generating fraction of the whole decimal is $\frac{7}{12}$, and the generating fraction of the recurring 3 is $\frac{4}{12}$, that is, $\frac{1}{3}$.

In the example $\frac{6}{35} = 1\dot{7}1428\dot{5}$, the generating fraction of the whole decimal is $\frac{6}{35}$, and the generating fraction of the recurring period 714285 is $\frac{25}{35}$, that is, $\frac{5}{7}$.

Attention has already been directed to the coincidence of the number of non-recurring digits in the decimal with the number of 2's or 5's in the denominator of the generating fraction of the decimal. The law exemplified in the examples referred to may be expressed thus:

When 2 *or* 5 *occurs among the factors of the denominator of the generating fraction in its lowest terms, the decimal generated is a mixed decimal, and the number of non-recurring digits in it is determined by the number of* 2*'s or* 5*'s in the denominator of the generating fraction; if* 2 *and* 5 *both occur, the number of non-recurring digits is determined by the one that occurs the more often in the denominator.*

It is left to the student to prove this law.

If the number of digits in a recurring period is even, we can obtain the digits of the second half by taking from 9 each digit of the first half, beginning with the left-hand digit. Thus:

$\frac{2}{19} = \cdot 105263157$, from which stage we can obtain the remaining digits by subtraction: $9 - 1 = 8$, $9 - 0 = 9$, $9 - 5 = 4$, $9 - 2 = 7$, $9 - 6 = 3$, $9 - 3 = 6$, $9 - 1 = 8$, $9 - 5 = 4$, $9 - 7 = 2$;

$$\therefore \ \frac{2}{19} = \dot{1}0526315789473684\dot{2}.$$

EXAMPLES. LI.

(*a*) Express as recurring decimals:

1. $\frac{2}{3}$.
2. $\frac{8}{9}$.
3. $\frac{5}{28}$.
4. $\frac{6}{7}$.
5. $\frac{3}{7}$.
6. $\frac{4}{13}$.
7. $\frac{7}{15}$.
8. $\frac{11}{14}$.
9. $\frac{17}{140}$.
10. $\frac{17}{65}$.
11. $\frac{1}{17}$.
12. $\frac{3}{19}$.

(*b*)

13. Reduce 10*s.* 4*d.* to the decimal of £1. 7*s.* 6*d.*

14. Express £2. 13*s.* 4½*d.* in decimal terms of 12½ guineas.

15. What decimal of 2 guineas is £1. 14*s.* 9¾*d.*?

16. What decimal part of £3. 15*s.* 9*d.* is equivalent to 7*s.* 2*d.*?

17. Express 1½ feet in decimal terms of ¼ mile.

18. Reduce 2·5 feet to the decimal of 1 yard.

19. What decimal part of 7 miles 4 furlongs is equivalent to 3 miles 6 furlongs 2 poles 2 yards 2 feet 3 inches?

20. Find the equivalent of 3 poles 5½ sq. yds. in decimal terms of an acre.

To find the generating fraction of a recurring decimal.

Preliminary:

$$\cdot 3 = \cdot 300\ldots;$$

$$\therefore \cdot 3 \times 10 = 3\cdot 00\ldots, \text{ that is, } 3.$$

$$\cdot\dot{3} = \cdot 333\ldots;$$

$$\therefore \cdot\dot{3} \times 10 = 3\cdot 33\ldots, \text{ that is, } 3\cdot\dot{3}.$$

$$\cdot 27 = \cdot 2700\ldots;$$

$$\therefore \cdot 27 \times 100 = 27\cdot 00\ldots, \text{ that is, } 27.$$

$$\cdot\dot{2}\dot{7} = \cdot 2727\ldots;$$

$$\therefore \cdot\dot{2}\dot{7} \times 100 = 27\cdot 27\ldots, \text{ that is, } 27\cdot\dot{2}\dot{7}.$$

The bearing of this will be seen in what follows.

Ex. 1. *Find the generating fraction of* $\cdot\dot{3}$.

Let x represent the required fraction; then

$$x = \cdot 33\ldots,$$

$$10x = 3\cdot 33\ldots;$$

$$\therefore 10x - x = 3\cdot 33\ldots - \cdot 33\ldots;$$

$$\therefore 9x = 3;$$

$$\therefore x = \tfrac{3}{9},$$

that is, $\frac{1}{3}$.

In the decimal there is one recurring digit; in the denominator of the fraction there is one 9.

Ex. 2. *Find the generating fraction of* $\cdot\dot{2}\dot{7}$.

Let x represent the required fraction; then

$$x = \cdot 2727\ldots,$$

$$100x = 27\cdot 2727\ldots;$$

$\therefore$ by subtraction,

$$99x = 27;$$

$$\therefore x = \tfrac{27}{99},$$

that is, $\frac{3}{11}$.

In the decimal there are two recurring digits; in the denominator of the fraction there are two 9's.

And always *a pure recurring decimal is equal to a vulgar fraction the numerator of which is the recurring period and the denominator as many 9's as there are digits in the recurring period.*

Ex. 3. *Find the generating fraction of* $\cdot\dot{8}5714\dot{2}$.

$$\text{The fraction}=\frac{857142}{999999};$$

the H.C.F. of the numerator and denominator is 142857;

$$\therefore\ \frac{857142}{999999}=\frac{142857\times6}{142857\times7}=\frac{6}{7};$$

$$\therefore\ \text{the required fraction is } \tfrac{6}{7}.$$

It follows from the rule that:

$$\cdot\dot{9}=1;\quad \cdot0\dot{9}=\cdot1;\quad \cdot4\dot{9}=\cdot5;\quad \cdot027\dot{9}=\cdot028;\quad \text{and so on.}$$

Ex. 4. *Find the vulgar fraction that is equivalent to* $\cdot4\dot{2}\dot{5}$.

Let x represent the required fraction,

$$\begin{aligned} \text{then}\qquad 10x&=\ \ \,4\cdot2525\ldots,\\ 1000x&=425\cdot2525\ldots\\ \hline \therefore\ \text{by subtraction,}\qquad 990x&=421;\\ \therefore\ x&=\tfrac{421}{990};\\ \therefore\ \cdot4\dot{2}\dot{5}&=\tfrac{421}{990}. \end{aligned}$$

We multiply by the power of 10 that will bring the non-recurring digit to the left of the decimal point, then by a power of 10 that will bring the first period to the left of the decimal point.

Or thus:

$$10x=4\cdot25\ldots,\ \text{that is, } 4\cdot\dot{2}\dot{5};$$

$$\text{the generating fraction of } \cdot\dot{2}\dot{5} \text{ is } \tfrac{25}{99};$$

$$\therefore\ 10x=4\tfrac{25}{99}=\tfrac{421}{99};$$

$$\therefore\ x=\tfrac{421}{990}.$$

The same result is obtained thus:

$$\frac{425-4}{990}=\frac{421}{990},$$

where we have subtracted the non-recurring digit from the whole decimal, and made the difference the numerator of a fraction having for its denominator a 9 for every recurring digit in the decimal and a 0 for the non-recurring digit.

Ex. 5. *Find the vulgar fraction that is equivalent to* $\cdot93\dot{1}\dot{8}$.

Let x represent the fraction,

$$\begin{aligned} \text{then}\quad 10000x&=9318\cdot18\ldots\\ \text{and}\quad 100x&=\ \ \,93\cdot18\ldots\\ \hline \therefore\ \text{by subtraction,}\quad 9900x&=9225;\\ \therefore\ x&=\tfrac{9225}{9900}=\tfrac{41}{44}. \end{aligned}$$

Or thus:

$$100x = 93\cdot 18 \ldots = 93\cdot\dot{1}\dot{8};$$

the generating fraction of $\dot{1}\dot{8}$ is $\frac{18}{99}$ or $\frac{2}{11}$;

$$\therefore\ 100x = 93\tfrac{2}{11} = \frac{1025}{11};$$

$$\therefore\ x = \frac{1025}{1100} = \frac{41}{44};$$

$$\therefore\ \cdot 93\dot{1}\dot{8} = \frac{41}{44}.$$

The same result is obtained thus, as in Ex. 4:

$$\frac{9318-93}{9900} = \frac{9225}{9900} = \frac{41}{44}.$$

From these examples we may deduce the following rule for finding the generating fraction of a *mixed recurring decimal*:

Subtract the non-recurring part from the whole decimal, make the difference the numerator of the fraction, and put for the denominator as many 9's as there are recurring digits and as many 0's as there are non-recurring digits in the decimal.

EXAMPLES. LII.

Find in their lowest terms the generating fractions of the following:

1. $\cdot\dot{3}$.
2. $\cdot\dot{5}\dot{4}$.
3. $\cdot\dot{1}0\dot{8}$.
4. $2\cdot\dot{1}\dot{7}$.
5. $7\cdot\dot{5}0\dot{4}$.
6. $\cdot 011\dot{3}\dot{6}$.
7. $\cdot 1\dot{2}\dot{7}$.
8. $\cdot 01\dot{0}\dot{9}$.
9. $\cdot 236\dot{1}$.
10. $3\cdot 064\dot{8}1\dot{4}$.
11. $\cdot 002\dot{6}$.
12. $\cdot 03\dot{2}\dot{8}$.
13. $5\cdot 0\dot{1}\dot{8}$.
14. $3\cdot\dot{0}1\dot{2}$.
15. $2\cdot\dot{0}0040\dot{7}$.
16. $\cdot 31\dot{4}36\dot{5}$.

If the fraction $\dfrac{1}{n}$ produces a pure recurring decimal, n is a measure of a number made up wholly of 9's.

How to utilise this law is shown below:

$$\tfrac{1}{3} = \tfrac{3}{9} = \cdot\dot{3};$$

$$\therefore\ \tfrac{2}{3} = \tfrac{6}{9} = \cdot\dot{6}.$$

$$\tfrac{1}{11} = \tfrac{9}{99} = \cdot\dot{0}\dot{9};$$

$$\therefore\ \tfrac{2}{11} = \tfrac{18}{99} = \cdot\dot{1}\dot{8},$$

$$\tfrac{5}{11} = \tfrac{45}{99} = \cdot\dot{4}\dot{5},$$

$$\tfrac{10}{11} = \tfrac{90}{99} = \cdot\dot{9}\dot{0}.$$

$$\tfrac{1}{37} = \tfrac{3}{111} = \tfrac{27}{999} = \cdot\dot{0}2\dot{7};$$

$$\therefore\ \tfrac{3}{37} = \tfrac{9}{111} = \tfrac{81}{999} = \cdot\dot{0}8\dot{1},$$

$$\tfrac{12}{37} = \tfrac{36}{111} = \tfrac{324}{999} = \cdot\dot{3}2\dot{4}.$$

The student should verify, and learn to identify at sight, any of the following equated forms:

$\frac{1}{3} = \cdot\dot{3}$,
$\frac{2}{3} = \cdot\dot{6}$.

$\frac{1}{6} = \cdot 1\dot{6}$,
$\frac{5}{6} = \cdot 8\dot{3}$.

$\frac{1}{9} = \cdot\dot{1}$,
$\frac{2}{9} = \cdot\dot{2}$,
$\frac{3}{9} = \cdot\dot{3}$,
$\frac{4}{9} = \cdot\dot{4}$,
$\frac{5}{9} = \cdot\dot{5}$,
$\frac{6}{9} = \cdot\dot{6}$,
$\frac{7}{9} = \cdot\dot{7}$,
$\frac{8}{9} = \cdot\dot{8}$,
$1 = \cdot\dot{9}$.

$\frac{1}{7} = \cdot\dot{1}4285\dot{7}$,
$\frac{2}{7} = \cdot\dot{2}8571\dot{4}$,
$\frac{3}{7} = \cdot\dot{4}2857\dot{1}$,
$\frac{4}{7} = \cdot\dot{5}7142\dot{8}$,
$\frac{5}{7} = \cdot\dot{7}1428\dot{5}$,
$\frac{6}{7} = \cdot\dot{8}5714\dot{2}$.

$\frac{1}{11} = \cdot\dot{0}\dot{9}$,
$\frac{2}{11} = \cdot\dot{1}\dot{8}$,
$\frac{3}{11} = \cdot\dot{2}\dot{7}$,
$\frac{4}{11} = \cdot\dot{3}\dot{6}$,
$\frac{5}{11} = \cdot\dot{4}\dot{5}$,
$\frac{6}{11} = \cdot\dot{5}\dot{4}$,
$\frac{7}{11} = \cdot\dot{6}\dot{3}$,
$\frac{8}{11} = \cdot\dot{7}\dot{2}$,
$\frac{9}{11} = \cdot\dot{8}\dot{1}$,
$\frac{10}{11} - \cdot\dot{9}\dot{0}$.

$\frac{1}{13} = \cdot\dot{0}7692\dot{3}$,
$\frac{2}{13} = \cdot\dot{1}5384\dot{6}$,
$\frac{3}{13} = \cdot\dot{2}3076\dot{9}$,
$\frac{4}{13} = \cdot\dot{3}0769\dot{2}$,
$\frac{5}{13} = \cdot\dot{3}8461\dot{5}$,
$\frac{6}{13} = \cdot\dot{4}6153\dot{8}$,
$\frac{7}{13} = \cdot\dot{5}3846\dot{1}$,
$\frac{8}{13} = \cdot\dot{6}1538\dot{4}$,
$\frac{9}{13} = \cdot\dot{6}9230\dot{7}$,
$\frac{10}{13} = \cdot\dot{7}6923\dot{0}$,
$\frac{11}{13} = \cdot\dot{8}4615\dot{3}$,
$\frac{12}{13} = \cdot\dot{9}2307\dot{6}$,

By using these results we can often greatly lessen the labour of finding the generating fractions of recurring decimals. Thus:

(1) $$\cdot 41\dot{6} = \cdot 4\tfrac{1}{6} = \frac{4\frac{1}{6}}{10} = \frac{25}{60} = \frac{5}{12}.$$

(2) $$\cdot 9\dot{5}\dot{4} = \cdot 9\tfrac{6}{11} = \frac{9\frac{6}{11}}{10} = \frac{105}{110} = \frac{21}{22}.$$

(3) $$\cdot 307\dot{3}0769\dot{2} = \cdot 307\tfrac{4}{13} = \frac{307\frac{4}{13}}{1000} = \frac{3995}{13000} = \frac{799}{2600}.$$

(4) $$3\cdot 3\dot{4}2857\dot{1} = 3\cdot 3\tfrac{3}{7} = 3\,\frac{3\frac{3}{7}}{10} = 3\tfrac{24}{70} = 3\tfrac{12}{35}.$$

If only approximate accuracy is needed, recurring decimals may be added, subtracted, multiplied, and divided by the methods of decimal approximation already explained; but, if exact results are required, we should use the methods exemplified below.

Ex. 1. *Find to five decimal places, and also exactly, the sum of*

$$2{\cdot}\dot{4}1\dot{8},\quad 1{\cdot}1\dot{6},\quad 3{\cdot}\dot{0}0\dot{9},\quad {\cdot}735\dot{4},\quad 24{\cdot}042.$$

Approximate.

$$\begin{array}{r|l}
2{\cdot}41841 & 84\\
1{\cdot}16666 & 66\\
3{\cdot}00900 & 90\\
{\cdot}73544 & 44\\
24{\cdot}042 & \\
\hline
31{\cdot}37154 &
\end{array}$$

We carry the recurring digits to the seventh place to make sure of the right carrying figure.

Exactly.

$$\begin{array}{r|l|l}
2{\cdot}418 & 418 & 41\\
1{\cdot}166 & 666 & 66\\
3{\cdot}009 & 009 & 00\\
{\cdot}735 & 444 & 44\\
24{\cdot}042 & & \\
\hline
31{\cdot}371 & 538 &
\end{array}$$

that is, $31{\cdot}371\dot{5}3\dot{8}$.

The number of digits in the non-recurring part of the answer is determined by the largest number of non-recurring digits in the addends. Here it is three; hence the recurring digits of the answer begin at the fourth place. We draw a vertical line to mark the beginning of the recurring period. The length of the period is determined by the L.C.M. of the numbers of digits contained in the periods of the addends, here 3, 1, 3, 1, the L.C.M. of which is 3. The recurring period of the answer will contain three digits, and therefore we extend the recurring digits three places, and two more to make sure of the right carrying figure. What is here said about addition applies also to subtraction.

Ex. 2. *Find to four decimal places, and also exactly, the difference between* $4{\cdot}1\dot{2}0\dot{3}$ *and* $2{\cdot}0\dot{6}\dot{1}$.

To four decimal places.

$$\begin{array}{r|l}
4{\cdot}1203 & 20\\
2{\cdot}0616 & 16\\
\hline
2{\cdot}0587 &
\end{array}$$

that is, $2{\cdot}0587$.

Exactly.

$$\begin{array}{r|l|l}
4{\cdot}1 & 203203 & 20\\
2{\cdot}0 & 616161 & 61\\
\hline
2{\cdot}0 & 587041 &
\end{array}$$

that is, $2{\cdot}0\dot{5}8704\dot{1}$.

Ex. 3. *Find to four decimal places, and also exactly, the product of* $5{\cdot}6\dot{1}4285\dot{7}$ *and* 9.

To four decimal places.

$$\begin{array}{r|l}
5{\cdot}6142 & 857\\
9 & \\
\hline
50{\cdot}5286 &
\end{array}$$

that is, $50{\cdot}5286$.

Exactly.

$$\begin{array}{r|l}
5{\cdot}6142857 & 14\\
9 & \\
\hline
50{\cdot}5285714 &
\end{array}$$

that is, $50{\cdot}5\dot{2}8571\dot{4}$.

Or thus (exactly):

$$5{\cdot}6\dot{1}4285\dot{7} = 5{\cdot}6\tfrac{1}{7} = 5\tfrac{43}{70};$$

$$\therefore\ 5{\cdot}6\dot{1}4285\dot{7} \times 9 = 5\tfrac{43}{70} \times 9 = 50\tfrac{37}{70} = 50{\cdot}5\dot{2}8571\dot{4}.$$

If the multiplicand and the multiplier both contain recurring decimals and the product is to be accurate, reduce the decimals to vulgar fractions, find the product, and express it in its decimal equivalent; but, if an approximate answer is required, the product can be obtained by the method explained and illustrated in connection with approximate products of finite decimal factors; and a similar remark applies to the division of recurring decimals by recurring decimals.

Ex. 4. *Find to five decimal places, and also exactly, the product of* $\cdot\dot{8}\dot{1} \times \cdot\dot{4}0\dot{7}$.

$$\cdot\dot{8}\dot{1} \times \cdot\dot{4}0\dot{7} = \cdot 818181 \ldots \times \cdot 407407 \ldots$$

$$= \cdot 0818181 \ldots \times 4\cdot 074074 \ldots .$$

To five decimal places.

$$\begin{array}{r|l} \cdot 08181 & 8181 \\ 47047 & 04 \\ \hline \cdot 32727 & 27 \\ 572 & 73 \\ 32 & 72 \\ & 57 \\ & 3 \\ \hline \cdot 33333 & 32 \end{array}$$

Note: the multiplier is reversed.

Exactly.

$$\begin{aligned} \cdot\dot{8}\dot{1} \times \dot{4}0\dot{7} &= \tfrac{81}{99} \times \tfrac{407}{999} \\ &= \tfrac{9}{11} \times \tfrac{11}{27} \\ &= \tfrac{1}{3} \\ &= \cdot\dot{3}. \end{aligned}$$

Ex. 5. *Divide to four decimal places, and also exactly,*

$$8\cdot 7\dot{6}\dot{7} \text{ by } 37\cdot\dot{5}\dot{7}.$$

To four decimal places.

$$\begin{array}{r} \cdot 23333 \\ \hline 37\cdot\cancel{5}\cancel{7}\cancel{5}\cancel{7}\,)\,8\cdot 76767 \\ 1\cdot 25253 \\ 12526 \\ 1253 \\ 126 \\ 13 \end{array}$$

Exactly.

$$\begin{aligned} 8\cdot 7\dot{6}\dot{7} \div 37\cdot\dot{5}\dot{7} &= \tfrac{8767-87}{990} \div \tfrac{3757-37}{99} \\ &= \tfrac{8680}{990} \times \tfrac{99}{3720} \\ &= \tfrac{8680}{990} \div \tfrac{3720}{99} \\ &= \tfrac{217}{930} = \cdot 2\dot{3}. \end{aligned}$$

EXAMPLES. LIII.

Find exactly the values of:

1. $\cdot 5\dot{7} + \cdot\dot{0}4\dot{3} + \cdot\dot{2}$.
2. $\cdot 4\dot{9} + \cdot\dot{0}4\dot{7} + \cdot 14\dot{6}\dot{3}$.
3. $\cdot\dot{0}7\dot{6} + \cdot 2\dot{4}\dot{5} + \cdot 20\dot{3}$.
4. $\cdot 1\dot{2}0\dot{3} - \cdot 0\dot{6}\dot{1}$.
5. $4\cdot\dot{1}23\dot{4} - \cdot 03\dot{7}4\dot{1}$.
6. $9\cdot 0\dot{3}11\dot{3} - 8\cdot 1\dot{9}\dot{3}$.
7. $9\cdot\dot{1}4285\dot{7} \times \cdot 109375$.
8. $\cdot 9\dot{2}8571\dot{4} \times \cdot\dot{3}0769\dot{2}$.

9. $\cdot 8\dot{3} \times \cdot\dot{3}8095\dot{2} \times \cdot 58\dot{3}$.

10. $\cdot\dot{8}\dot{1} \times \cdot\dot{4}0\dot{7} \times \cdot\dot{1}5384\dot{6}$.

11. $\cdot\dot{5}7142\dot{8} \times \cdot\dot{4}\dot{5} \times \cdot 58\dot{3}$.

12. $1\cdot 375 \times 2.\dot{3} \times \cdot\dot{2}\dot{7}$.

13. $4\cdot 2\dot{6} \div \cdot 7\dot{3}$.

14. $\cdot\dot{0}\dot{4} \div \cdot\dot{7}6923\dot{0}$.

15. $\cdot 1\dot{6} \div \cdot 00\dot{2}\dot{7}$.

16. $1\cdot 89 \div \cdot\dot{7}1428\dot{5}$.

17. Find the value of $\cdot 8\dot{3} \times \cdot\dot{3}8095\dot{2} \times \cdot 58\dot{3}$, retaining 3 decimal places.

18. Find the value of $\cdot 9\dot{2}8571\dot{4} \times \cdot\dot{3}0769\dot{2}$, retaining 5 decimal places.

19. Find the value of $\cdot\dot{8}\dot{1} \times \cdot\dot{4}0\dot{7} \times \cdot\dot{1}5384\dot{6}$, retaining 5 decimal places.

20. Find the value of $\cdot\dot{5}7142\dot{8} \times \cdot\dot{4}\dot{5} \times \cdot 58\dot{3}$, retaining 4 decimal places.

21. Divide $\cdot 23\dot{7}\dot{2}$ by $\cdot 177\dot{2}$, retaining 5 places of decimals.

22. Divide $\cdot\dot{8}54\dot{7}$ by $1\cdot 3\dot{6}76\dot{5}$, retaining 3 places of decimals.

23. Divide $91\cdot\dot{8}6\dot{3}$ by $87\cdot 5\dot{6}$, retaining 5 places of decimals.

24. Divide $12\cdot 9\dot{7}6923\dot{0}$ by $\cdot 0\dot{8}5714\dot{2}$, retaining 3 places of decimals.

MISCELLANEOUS EXAMPLES. LIV.

1. Simplify $\frac{3\cdot 25}{5\cdot 5} \times \frac{341}{3\cdot 125} \times \frac{6\cdot 875}{\cdot 341}$.

2. By what number must ·4485 be divided that the quotient may be ·0069?

3. Reduce $\frac{\cdot 0193 - \cdot 0155 + \cdot 0007}{1 - \cdot 2345 - \cdot 0455}$ to a vulgar fraction.

4. Reduce $\frac{1\cdot 003 + 1\cdot 9 - \cdot 05}{7 - \cdot 35 + \cdot 007}$ to a vulgar fraction.

5. Reduce $\frac{\cdot 03 - \cdot 02997 + \cdot 00003}{\cdot 00539 + \cdot 00048 - \cdot 00491}$ to a vulgar fraction.

6. Reduce $\frac{\cdot 47 - (\cdot 5 - \cdot 0303)}{\cdot 0873 - (\cdot 0083 + \cdot 06)}$ to a vulgar fraction.

7. Simplify $\frac{5\frac{1}{3} - \cdot 042 - 2\cdot 4 + 7\frac{5}{6}}{16\frac{2}{15} \div 60\frac{1}{2}}$, expressing the result as a decimal.

8. Simplify $\frac{8\cdot 064}{\cdot 84\dot{6} + \frac{20}{7} \text{ of } \cdot 291\dot{6}}$, expressing the result as a decimal.

9. Express the difference between $2\cdot\dot{5}3\dot{5}$ and 2·535 (1) by a vulgar fraction, and (2) by a circulating decimal.

10. Divide 100 into two parts such that one shall exceed the other by 8·75.

11. How many times can a cup containing ·013 of a pint be filled from a jug containing a quart, and what decimal of a pint will remain in the jug after the last filling of the cup?

12. How often is ·35 of an inch contained in 12·52 inches, and what is the value of the remainder?

13. *A* owns ·58 of a ship, and *B* the remainder: if the difference in the value of their shares is £37·46, what is the value of the ship?

14. The water of a town contains ·002, ·003, and ·02 of certain impurities: find what decimal part of the water is pure.

15. An imperial gallon contains 277·274 cubic inches: find to 3 decimals the number of gallons in a cubic foot.

16. A cubic metre of water contains 1000 litres, and a cubic foot of water $6\frac{1}{4}$ gallons: taking a linear metre as 3·28 feet, express to 4 decimal places the value of a litre as a decimal of a gallon.

17. Divide 18·2 into four such parts that the first shall exceed the second by ·2, the third by 3·6, and the fourth by 4·8.

18. *A* can walk $4{\cdot}1\dot{6}$ miles an hour, and *B* 4·5 miles an hour: find the ratio of their rates.

19. *A* walks 25 miles in 5·15 hours, and *B* 15 miles in 4·75 hours: what is the ratio of their rates?

20. Find the number of which ·21 is the square root.

21. Express in decimals the value of $\left(\frac{3}{4}\right)^3$.

22. Write the following decimals in equivalent standard form: ·0734 and ·000645.

23. Find by factors the square root of 11025.

24. Reduce $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ to fractions having a common denominator.

25. Reduce $\frac{14x^3+21x^2+7ax}{35x^2}$ to its lowest terms.

26. Find the value of x in $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=10$.

27. What factor must 2205 be multiplied by to make it a perfect square?

CHAPTER XII.

LOGARITHMS.

THE index of the power to which a number (called the *base*) must be raised to produce another number is called the *logarithm* of the other number. Thus: if $n = a^x$, then x is called the logarithm of n to the base a.

In the language of indices, n is the power, a the base, and x the index; and the notation is $n = a^x$.

In the language of logarithms, n is the number, a the base, and x the logarithm; and the notation is $\log_a n = x$, where log is the usual contraction for logarithm.

It will be seen that the expression $n = a^x$ and $\log_a n = x$ both denote that $n = a$ raised to the x^{th} power.

The generalised expression $\log_a n = x$ indicates that any number may serve as the base of a system of logarithms. Thus:

$$\log_4 16 = 2, \text{ for } 16 = 4^2;$$
$$\log_5 125 = 3, \text{ for } 125 = 5^3;$$
$$\log_7 2401 = 4, \text{ for } 2401 = 7^4; \text{ and so on.}$$

In the system of logarithms in common use, the base is 10, the base of our system of notation. Hence logarithms to this base are called *common logarithms*. When using them, we need not state the base. Thus:

$$\log_{10} 100 \text{ is written } \log 100,$$
$$\log_{10} 70 \text{ is written } \log 74; \text{ and so on.}$$

From the notation we have:

$$\log 1 = 0, \quad \text{for } 10^0 = 1;$$
$$\log 10 = 1, \quad \text{for } 10 = 10^1;$$
$$\log 100 = 2, \quad \text{for } 100 = 10^2;$$
$$\log 1000 = 3, \quad \text{for } 1000 = 10^3; \text{ and so on.}$$

$$\log \cdot 1 = -1, \text{ for } \cdot 1 = 10^{-1};$$
$$\log \cdot 01 = -2, \text{ for } \cdot 01 = 10^{-2};$$
$$\log \cdot 001 = -3, \text{ for } \cdot 001 = 10^{-3}; \text{ and so on.}$$

Hence the common logarithms of numbers between 10 and 100 lie between 1 and 2, the indices of 10 and 100 to the base 10, that is, they are 1 + a fraction; the logarithms of numbers between 100 and 1000 lie between 2 and 3, the indices of 100 and 1000, that is, they are 2 + a fraction; the logarithms of numbers between 1 and 10 are a fraction of 1; the logarithms of numbers between ·1 and 1 are not so small as − 1, and are represented by − 1 + a *positive fraction*; the logarithms of numbers between ·1 and ·01 are not so small as − 2, and are represented by − 2 + a *positive fraction*; the numbers between ·01 and ·001 are not so small as − 3, and are represented by − 3 + a *positive fraction.* The fraction is always expressed as a decimal.

That is, the common log of a number between

100 and	1000	(a number with 3 digits)	2 + a decimal,
10 and	100	(a number with 2 digits)	1 + a decimal,
1 and	10	(a number with 1 digit)	0 + a decimal,
·1 and	1		− 1 + a decimal,
·01 and	·1		− 2 + a decimal,
·001 and	·01		− 3 + a decimal.

The integral part of a logarithm is called the *characteristic,* and the decimal part the *mantissa.*

Negative characteristics are written with the sign − above them, as $\bar{1}$, $\bar{2}$, $\bar{3}$.

From the foregoing table, it is obvious:

(1) that the characteristic of the common logarithm of any whole or mixed number is *less by one than the number of integral figures in the corresponding number*, that is, it is equal to the number of digits on the left of the units' digit, which is regarded as the zero of reckoning;

(2) that, though the characteristic of the logarithm of a decimal fraction is *negative,* the mantissa or decimal part of the logarithm is *positive*;

(3) that, in decimal fractions, if there is no 0 between the decimal point and the first significant figure of the decimal,

the characteristic of the log is $\bar{1}$; if there is one 0 between them, the characteristic is $\bar{2}$; if there are two 0's between them, the characteristic is $\bar{3}$, and so on; that is, to find the characteristic of a decimal, count from the units' place as zero to the first significant figure of the decimal.

(1) and (3) may be expressed thus: *The characteristic of the common logarithm of a number is identical with the order of the highest significant figure of the number.*

NOTE. The significant figures of a number are, as we have seen, the figures that remain when the zeros at the beginning or end have been removed. Thus, the significant figures of the numbers 743500 and ·007435 are the same, for they are 7435.

The mantissae of the common logarithms of all numbers consisting of the same significant figures placed in the same order are the same, whether the numbers be whole or fractional or mixed. For example:

$\log 743500 = 5{\cdot}8713$

$\log 74350 = 4{\cdot}8713$

$\log 7435 = 3{\cdot}8713$

$\log 743{\cdot}5 = 2{\cdot}8713$

$\log 74{\cdot}35 = 1{\cdot}8713$

$\log 7{\cdot}435 = 0{\cdot}8713$

$\log {\cdot}7435 = \bar{1}{\cdot}8713$

$\log {\cdot}07435 = \bar{2}{\cdot}8713$

$\log {\cdot}007435 = \bar{3}{\cdot}8713.$

That is, $743500 = 10^{5{\cdot}8713}$ (approx.)

$74{\cdot}35 = 10^{1{\cdot}8713}$ (approx.), and so on.

The mantissa of the logarithms given above is taken from a four-figure table, a table that gives mantissae to four decimal places. Four-place tables give results sufficiently accurate for ordinary computations, that is, results correct to four significant figures.

Being indices of the same base, logarithms are subject to the laws of indices; and, as we have seen,

$$a^m \times a^n = a^{m+n}, \text{ and } a^m \div a^n = a^{m-n}.$$

Hence

(1) *the logarithm of a product = the sum of the logarithms of the factors of the product; and* (2) *the logarithm of a quotient = the algebraic difference of the logarithms of the dividend and the divisor.*

Thus:

(1) Given that $\log 75 = 1{\cdot}8751$, and $\log 85 = 1{\cdot}9294$,

$$\text{then} \quad \log(75 \times 85) = 1{\cdot}8751 + 1{\cdot}9294$$
$$= 3{\cdot}8045.$$

Otherwise:
$$75 = 10^{1{\cdot}8751},$$
$$85 = 10^{1{\cdot}9294};$$
$$\therefore\ 75 \times 85 = 10^{1{\cdot}8751} \times 10^{1{\cdot}9294} = 10^{3{\cdot}8045};$$
$$\therefore\ \log(75 \times 85) = 3{\cdot}8045.$$

(2) From the data in (1),

$$\log(85 \div 75) = 1{\cdot}9294 - 1{\cdot}8751$$
$$= 0{\cdot}0543,$$

$$\text{and generally} \quad \log ab = \log a + \log b,$$

$$\text{and} \quad \log \frac{a}{b} = \log a - \log b.$$

We shall use multiplication and division to find the logarithms of 743500 and ·007435, given that $\log 7{\cdot}435 = {\cdot}8713$.

$$743500 = 7435 \times 10^2;$$
$$\therefore\ \log 743500 = \log 7435 + \log 10^2$$
$$= 3{\cdot}8713 + 2$$
$$= 5{\cdot}8713,$$
$${\cdot}007435 = 7435 \times 10^{-6}$$
$$= \frac{7435}{10^6};$$
$$\therefore\ \log {\cdot}007435 = \log 7435 - \log 10^6$$
$$= 3{\cdot}8713 - 6$$
$$= \overline{3}{\cdot}8713.$$

EXAMPLES. LV.

1. Given that log 36 = 1·5563, find the logarithms of

360, 36000, ·036, ·00036, ·0000036.

2. Given that log 5 = ·6990, and log 7 = ·8451, find the logarithms of 35, 3500, ·0035, ·000035.

3. Given that log 8 = ·9031, and log 9 = ·9542, find the logarithms of 72, 72000, ·0072, ·0000072.

4. Given that log 2 = ·3010, log 3 = ·4771, log 7 = ·8451, log 11 = 1·0414, calculate the values of the following:

(*a*) log 21, (*b*) log 14, (*c*) log 77, (*d*) log 33,

(*e*) log $\frac{11}{3}$, (*f*) log $3\frac{1}{2}$, (*g*) log $1\frac{4}{7}$.

Logarithmic Tables and how to use them.

Some tables give mantissae to 4 decimal places, some to 5, some to 6, and some to 7. It is customary to use tables giving mantissae to a number of places equal to the number of significant figures required in the computation to be made. Obviously, the larger the number of places in the mantissa of a logarithm, the nearer to accuracy are the results obtained from the use of it.

The following is an extract from a table of four-figure logarithms. Only the mantissae of the logarithms are given, and they are printed without the decimal point.

Logarithms.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13

In the first column, called the number column, are the numbers 26, 27, 28, 29, 30; and in line with 26, and in the column headed 0, is the mantissa ·4150, which is the mantissa for the sequence of digits 26. The characteristic of 26 is 1;

$$\therefore \log 26 = 1{\cdot}4150.$$

In line with 27, and in the column headed 0, is the mantissa ·4314, which is the mantissa for the sequence of digits 27. The characteristic of 27 is 1;

$$\therefore \log 27 = 1{\cdot}4314.$$

Similarly, $\log 28 = 1{\cdot}4472$,

$\log 29 = 1{\cdot}4624$,

$\log 30 = 1{\cdot}4771$.

When a number consists of three figures, the first two are found in the number column, and the last is found in one of the columns headed 1, 2, 3, 4, 5, 6, 7, 8, 9; and, in line with the two first and in the column headed by the last, is found the mantissa for the sequence of digits in the three-figure number. Thus, the mantissa ·4216 for the sequence 264 is found in line with 26, and in the column headed 4; and the mantissa ·4609 for the sequence 289 is found in line with 28, and in the column headed 9.

The characteristic of 264 is 2;

$$\therefore \log 264 = 2{\cdot}4216.$$

The characteristic of 289 is 2;

$$\therefore \log 289 = 2{\cdot}4609.$$

Similarly, $\log 295 = 2{\cdot}4698$,

$\log 308 = 2{\cdot}4886$; and so on.

On the right of the table are columns headed 1, 2, 3 ... 9. These are called "difference columns," and their use is to enable us to find what must be added to the mantissa for a three-figure number to obtain the mantissa for a four-figure number, the first three digits of which are the same and have the same sequence as the digits of the three-figure number.

Suppose we have to find the logarithms of 2647 and 2895. The mantissae for 264 and 289 are ·4216 and ·4609. Under 7 of the difference columns we find 11, which means ·0011. Adding this to ·4216, we have ·4227, which is the mantissa for the sequence of digits 2647.

The characteristic of 2647 is 3;

$$\therefore \log 2647 = 3{\cdot}4227.$$

Again, the mantissa for the sequence of digits 289 is ·4609. Under 5 of the difference columns we find 8, which means ·0008. Adding this to ·4609, we have ·4617, which is the mantissa for the sequence of digits 2895.

The characteristic of 2895 is 3;

$\therefore$ log 2895 = 3·4617.

Similarly, log 2764 = 3·4415,

log 3026 = 3·4809; and so on.

Since numbers that have the same significant figures in the same order have in common logarithms the same mantissae, the mantissae for 9, 8, 7, 6, and so on, are the same as those for 90, 80, 70, and so on.

EXAMPLES. LVI.

Find the common logarithms of:

1. 271, 2·71, ·271, ·00271.

2. 2934, 29·34, ·2934, ·0002934.

3. 30, 3, ·03, ·000003.

By inverting the process of finding the log of a given number, we can readily find the number that corresponds to a given log, that is, the *antilogarithm* of the log. Take log 1·4814; search for the row of mantissae containing 4814. We find it in the row beginning with 30, and in the column headed 3; hence the sequence of digits corresponding to the mantissa 4814 is 303; and, since the characteristic of the log is 1, the number sought is 30·3.

Suppose we have to find the antilog of a log the mantissa of which does not occur in the table, for instance, of log 2·4860.

Searching the table, we find in the row beginning with 30, and in the column headed 6, the mantissa 4857, which is 3 (really ·0003) less than the mantissa 4860. Carrying the eye along the same row to the difference column, we find 3 in the column headed 2. 2 is therefore the next digit of the antilog sought. The characteristic of the log is 2; hence the antilog of log 2·4860 is 306·2.

Or we may find the antilog thus:

(1) Take the difference between the mantissa 4857 and the mantissa 4860, namely, 3;

(2) take the difference between the mantissa 4857 and the next larger mantissa 4871 in the same row, namely, 14;

(3) take the difference between the antilogs of the mantissae 4857 and 4871, that is, the difference between 306 and 307, namely, 1; then the antilog sought lies $\frac{3}{14}$ of 1 of the way

between 306 and 307. $\frac{3}{14}$ of $1 = \cdot 2$ (approx.); hence the sequence of digits in the antilog sought is 3062. The characteristic of the given log is 2; therefore the antilog sought is 306·2, which is the antilog we got from the table.

If the difference between the mantissa of the table and the mantissa of the given log is not found in the table of differences, choose the nearest, taking the larger if the number is equidistant from two numbers found in these columns.

The student may defer finding antilogs until he has read the next article. He will then be able to check his results by referring them to the table of antilogarithms given with the table of logarithms at the end of the book.

To save time and trouble, tables have been constructed from which we can read off numbers of which the logarithms are known. Such tables are called *antilogarithm tables.* The following is an extract from one.

Antilogarithms.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4

In the first column of the table are given the first and second figures of mantissae, the third figure being found at the head of one of the columns 0 to 9, and the fourth at the head of one of the difference columns on the right. The four-figure numbers in the columns are the *significant figures* of the antilogarithms, that is, of the natural numbers corresponding to the mantissae of the logarithms.

Ex. 1. *Find the number corresponding to* log 1·2745.

Find ·27 in the mantissa column, and look along the row to the column headed 4, where we find 1879, which is the digit-sequence for the mantissa ·2740. In the same row take the number in the difference column headed 5. The number is 2;

$\therefore$ the significant figures of the required number $= 1879 + 2 = 1881$.

Since the characteristic of the logarithm is 1, there are two figures in the integral part of the required number;

$\therefore$ the number $= 18 \cdot 81$.

It follows that the antilogarithm of log ·2745 = 1·881,

the antilogarithm of log $\bar{1}$·2745 = ·1881,

the antilogarithm of log $\bar{2}$·2745 = ·01881,

the antilogarithm of log $\bar{3}$·2745 = ·001881,

the antilogarithm of log 2·2745 = 188·1,

the antilogarithm of log 3·2745 = 1881.

Ex. 2. *Find the number corresponding to log* $\bar{1}$·3068.

From the table we find that the significant figures for the mantissa ·3060 are 2023, and that the difference for the 8 of the given mantissa is 4;

∴ the significant figures of the required number are 2027.

The characteristic of the logarithm is $\bar{1}$; hence the significant figures of the required number begin at the first decimal place;

∴ the required number is ·2027.

EXAMPLES. LVII.

Find the numbers corresponding to the following logarithms:

1. ·2632, 1·2632, 2·2632, $\bar{1}$·2632.

2. ·2749, 1·2749, 2·2749, $\bar{2}$·2749.

3. ·2953, 1·2953, 3·2953, $\bar{3}$·2953.

Multiplication by means of Logarithms.

For the work that follows, the Tables given at the end of the book must be consulted.

To find the product of numbers by means of their logarithms, we find the log of the multiplier and of the multiplicand, and add them for the log of the product; for

$$\log a + \log b = \log ab.$$

Ex. 1. *The top of a table is* 3·75 *metres long and* 2·875 *metres wide: find its area.*

log 3·75 = ·5740
log 2·875 = ·4587

Sum of logs = 1·0327
= log of product;

∴ product = antilog 1·0327
= 10·78,

that is, area ≑ 10·78 sq. metres (approx.).

The exact product of 3·75 × 2·875 is 10·78125.

Error; ·00125.

Ex. 2. *Multiply* 73·47 *by* ·0625.

$$\begin{aligned} \log 73{\cdot}47 &= 1{\cdot}8661 \\ \log {\cdot}0625 &= \underline{\bar{2}{\cdot}7959} \\ \text{Sum of logs} &= 0{\cdot}6620\,; \end{aligned}$$

$\therefore$ antilog of sum $= 4{\cdot}592$,

that is, product $= 4{\cdot}592$ (approx.).

The exact product of 73·47 × ·0625 is 4·591875.

Error : ·000125.

NOTE 1. As mantissae are always positive, numbers carried from their addition are always positive; hence, in the foregoing addition, we have

$$1\,(\text{carried}) + (-2) + 1 = 2 - 2 = 0.$$

NOTE 2. Computations with negative characteristics are made according to the rules of algebraic addition and subtraction; hence, to add a positive and a negative characteristic, we take their difference, and make it positive or negative according as the positive or the negative characteristic is the greater, as

$$6 + \bar{2} = 4\,;\ \bar{6} + 2 = \bar{4}\,;$$

and, to subtract a negative characteristic, we change its sign to plus, and then proceed as in addition; thus,

$$\text{the difference between } 2 \text{ and } \bar{4} = 2 + 4 = 6,$$

$$\text{for } 2 - (-4) = 2 + 4 = 6\,;$$

$$\text{the difference between } \bar{2} \text{ and } \bar{4} = -2 + 4 = 2,$$

$$\text{for } -2 - (-4) = -2 + 4 = 2.$$

Ex. 3. *Multiply* ·2594 *by* ·0536.

$$\begin{aligned} \log {\cdot}2594 &= \bar{1}{\cdot}4140 \\ \log {\cdot}0536 &= \underline{\bar{2}{\cdot}7292} \\ \text{sum of logs} &= \bar{2}{\cdot}1432 \end{aligned}$$

antilog of sum $= {\cdot}01391$,

that is, product $= {\cdot}0139$ correct to 4 places.

The sum of the characteristics :

$$1\,(\text{carried}) + (-2) + (-1) = -2.$$

The exact product is ·01390384.

Error : ·00000616.

It must be clearly understood that in work with four-figure logarithms the results obtained are only approximate; they are true to three significant figures; and, as we have seen above, the fourth figure, though not necessarily accurate, is not far from accuracy. When a closer approximation to accuracy is required, five-, six-, or seven-figure logarithms should be used.

EXAMPLES. LVIII.

1. 49·57 × 2·565.

2. 64·23 × 73·64 × ·0375.

3. A rectangular garden is 67·58 metres long and 42·37 metres wide: find its area.

4. Find the weight of 19·72 cubic feet of iron, given that 1 cubic foot weighs 465 lbs.

5. Find the cubic content of a rectangular block of stone 2·234 metres long, ·736 metres wide, and 1·36 metres deep.

6. A piece of carpet is 3·76 metres long and ·75 metres wide: what is its area?

Division by means of Logarithms.

We have seen that $\log\frac{a}{b} = \log a - \log b$, that is, *that the logarithm of a quotient = the logarithm of the dividend − the logarithm of the divisor.*

Ex. 1. *Divide* 8568 *by* 547.

log 8568 = 3·9329
log 547 = 2·7380
Difference of logs = 1·1949
Antilog 1·1949 = 15·66,
that is, quotient = 15·66.

By division the quotient is 15·6636....

Error: ·0036....

Ex. 2. *Divide* 547 *by* 8568.

log 547 = 2·7380
log 8568 = 3·9329
Difference of logs = $\bar{2}$·8051
Antilog $\bar{2}$·8051 = ·06384,
that is, quotient = ·06384.

By division the error is found to be less than ·000005.

Subtraction of the characteristics or indices:

2 − 1 (carried) − 3 = −2.

Ex. 3. *Divide* ·8568 *by* 547.

log ·8568 = $\bar{1}$·9329
log 547 = 2·7380
Difference of logs = $\bar{3}$·1949
Antilog $\bar{3}$·1949 = ·001566,
that is, quotient = ·001566,

Subtraction of the characteristics or indices:

−1 − (+2) = −3.

Ex. 4. *Divide* ·00547 *by* ·8568.

log ·00547 = $\bar{3}$·7380
log ·8568 = $\bar{1}$·9329
Difference of logs = $\bar{3}$·8051
Antilog $\bar{3}$·8051 = ·006384

Subtraction of the characteristics or indices:

−3 − (−1) − 1 (carried)
= −3 + 1 − 1 = −3.

Ex. 5. *Find the value of* $22{\cdot}4 \times \frac{273}{285{\cdot}6} \times \frac{726{\cdot}2}{760}$.

$$\text{Let } x = 22{\cdot}4 \times \frac{273}{285{\cdot}6} \times \frac{726{\cdot}2}{760};$$

then $\log x = \log 22{\cdot}4 + \log 273 + \log 726{\cdot}2 - \log 285{\cdot}6 - \log 760$.

$$\begin{aligned} \log 22{\cdot}4 &= 1{\cdot}3502 \\ \log 273 &= 2{\cdot}4362 \\ \log 726{\cdot}2 &= \underline{2{\cdot}8610} \end{aligned} \qquad \begin{aligned} \log 285{\cdot}6 &= 2{\cdot}4557 \\ \log 760 &= \underline{2{\cdot}8808} \\ \text{sum of logs} &= 5{\cdot}3365 \end{aligned}$$

$$\begin{aligned} \text{Sum of logs of dividend} &= 6{\cdot}6474 \\ \text{Sum of logs of divisor} &= \underline{5{\cdot}3365} \\ \therefore \log x &= 1{\cdot}3109 \end{aligned}$$

$$\begin{aligned} \text{Antilog } 1{\cdot}3109 &= 20{\cdot}46, \\ \text{that is, } x &= 20{\cdot}46. \end{aligned}$$

EXAMPLES. LIX.

1. Find the value of $\frac{{\cdot}042}{{\cdot}001833}$.

2. Find the value of $\frac{38{\cdot}56}{271{\cdot}4} \times \frac{4{\cdot}381}{73{\cdot}45}$.

3. The length of the circumference of a circle is approximately $\frac{355}{113}$ times the length of its diameter: find the length of the circumference of a circle the diameter of which is 15·67 metres.

4. If $a = 17{\cdot}28$, $b = {\cdot}347$, and $c = 2{\cdot}685$, find the value of x in the equation $x = \frac{ab}{c}$.

Involution and Evolution by means of Logarithms.

$$(10^2)^2 = 10^2 \times 10^2 = 10^{2+2} = 10^{2\times 2};$$

$$(10^2)^3 = 10^2 \times 10^2 \times 10^2 = 10^{2+2+2} = 10^{2\times 3};$$

and generally, $(a^m)^n = a^{mn}$.

Let $\log m = x$ (to base 10),

then $m = 10^x$;

$$\begin{aligned} \therefore\ m^n &= (10^x)^n \\ &= 10^{xn}; \\ \therefore\ \log m^n &= xn \\ &= n \log m \quad (\text{for } x = \log m). \end{aligned}$$

Or thus:

$$\text{Let} \quad x = a \times a \times a \times a = a^4;$$

then, since the logarithm of a product is equal to the sum of the logarithms of its factors,

$$\begin{aligned} \therefore \log x &= \log a + \log a + \log a + \log a \\ &= 4 \log a. \end{aligned}$$

Hence the logarithm of any power of a number is equal to the logarithm of the number multiplied by the index of the power.

Ex. 1. *Find the value of* $(6{\cdot}395)^4$.

$$\begin{array}{rr} \log 6{\cdot}395 = & {\cdot}8058 \\ & 4 \\ \hline \log (6{\cdot}395)^4 = & 3{\cdot}2232 \end{array}$$

antilog = 1672 approx. (Error less than 1).

Ex. 2. *Find the cube of* ·7689.

$$\begin{array}{rr} \log {\cdot}7689 = & \bar{1}{\cdot}8859 \\ & 3 \\ \hline \log ({\cdot}7689)^3 = & \bar{1}{\cdot}6577 \end{array}$$

antilog = ·4546 approx. (Error less than ·0001).

The number carried from the mantissa is always positive; 2 is carried from the mantissa of Ex. 2, and the difference between -1×3 and 2 is $-3 + 2$, that is, -1.

In the involution of integral numbers by logarithms the actual error may be large; but, in relation to the product, it is always small, as will be seen from the next example.

Ex. 3. *Find the cube of* 365.

$$\begin{array}{rr} \log 365 = & 2{\cdot}5623 \\ & 3 \\ \hline \log (365)^3 = & 7{\cdot}6869 \end{array}$$

antilog = 48,630,000 approx.

The actual error is 2875, which is an error of 2875 in 48,627,125 or less than 1 in 16,000, a very near approximation to accuracy.

$$\sqrt{79} = (79)^{\frac{1}{2}}; \quad \sqrt[3]{79} = (79)^{\frac{1}{3}}; \quad \sqrt[4]{79} = (79)^{\frac{1}{4}};$$

$$\text{and generally } \sqrt[n]{a} = a^{\frac{1}{n}},$$

$$\text{and } \sqrt[n]{a^x} = a^{\frac{x}{n}}.$$

Hence $\log\sqrt{79} = \frac{1}{2}\log 79,\qquad \log\sqrt[3]{79} = \frac{1}{3}\log 79,$

$$\log\sqrt[4]{79} = \frac{1}{4}\log 79,\qquad \log\sqrt[n]{a} = \frac{1}{n}\log a,$$

$$\log\sqrt[n]{a^x} = \frac{x}{n}\log a.$$

Ex. 1. *Find the value of* $\sqrt{79}$.

Let x = the square root of 79,

then $x = (79)^{\frac{1}{2}}$;

$\therefore\ \log x = \frac{1}{2}\log 79$

$= \frac{1}{2}$ of $1{\cdot}8976$

$= {\cdot}9488$;

antilog $= 8888$;

$\therefore\ x = 8{\cdot}888$, correct to 3 decimal places.

Ex. 2. *Find the value of* $\sqrt[3]{79}$.

Let $x = \sqrt[3]{79}$,

then $x = (79)^{\frac{1}{3}}$;

$\therefore\ \log x = \frac{1}{3}\log 79$

$= \frac{1}{3}$ of $1{\cdot}8976$

$= {\cdot}6325$;

antilog $= 4290$;

$\therefore\ x = 4{\cdot}29$ approx.

Ex. 3. *Find the value of* $\sqrt[3]{(79)^2}$.

Let $x = \sqrt[3]{(79)^2}$,

then $x = (79)^{\frac{2}{3}}$;

$\therefore\ \log x = \frac{2}{3}\log 79$

$= \frac{2}{3}$ of $1{\cdot}8976$

$= 1{\cdot}2651$;

antilog $= 1841$;

$\therefore\ x = 18{\cdot}41$ approx.

The division of logarithms having negative characteristics.

When the characteristic is exactly divisible by the divisor, we write down the quotient with the negative sign, and divide the decimal part in the usual way, as

$$\bar{4}{\cdot}3768 \div 2 = \bar{2}{\cdot}1884;$$

but, when the characteristic is not exactly divisible by the divisor, we increase it by such a negative number as will make it divisible by the divisor; and, to neutralise this increase of the minus characteristic, we increase the positive part of the logarithm by prefixing to it the same digit, as

$$\bar{4}{\cdot}3768 \div 3 = \tfrac{1}{3} \text{ of } (\bar{6} + 2{\cdot}3768)$$
$$= \bar{2}{\cdot}7923.$$

This device has frequently to be adopted in the extraction of roots of fractional quantities.

Ex. 4. *Find the cube root of* ·01657.

$$\begin{aligned} \text{Let}\quad x &= \sqrt[3]{(\cdot01657)},\\ \text{then}\quad x &= (\cdot01657)^{\frac{1}{3}};\\ \therefore\ \log x &= \tfrac{1}{3}\log \cdot01657\\ &= \tfrac{1}{3}\text{ of }\bar{2}\cdot2193\\ &= \tfrac{1}{3}\text{ of }(\bar{3}+1\cdot2193)\\ &= \bar{1}\cdot4064;\\ \therefore\ x &= \cdot2549. \end{aligned}$$

EXAMPLES. LX.

1. Multiply $\bar{1}\cdot4183$ by 9, 12, 25.

2. Multiply $\bar{2}\cdot9117$ by 3, 5, 8.

3. Divide $\bar{3}\cdot9025$ by 4, 5, 6.

4. Divide $\bar{2}\cdot9581$ by 3, 4, 5, 6.

EXAMPLES. LXI.

Evaluate the following:

1. $\sqrt{2689}$.

2. $\sqrt{4978}$.

3. $\sqrt[3]{4367}$.

4. $\sqrt[3]{7598}$.

5. $\sqrt[5]{9547}$.

6. $\sqrt{\cdot0538}$.

7. $\sqrt[3]{\cdot00735}$.

8. $\sqrt[3]{\cdot0681}$.

9. $\sqrt[5]{\cdot00049}$.

10. $19\sqrt{586}$.

11. $3\sqrt{7}\times 23\sqrt[3]{\cdot039}$.

12. $\dfrac{17\sqrt{70\cdot23}}{15\sqrt[3]{21\cdot67}}$

13. $\dfrac{25\sqrt{35\cdot74}}{16\sqrt[3]{2\cdot75}}$

14. $\dfrac{29\sqrt{\cdot065}}{14\sqrt[4]{\cdot007}}$

[If the edge of a cube $= x$ inches, the content of cube $= x^3$ cubic inches, and edge of cube $= \sqrt[3]{x^3}$ inches.]

15. Find the edge of a cubic block of metal containing 78·54 cubic inches.

16. The edge of a cubical block of wood measures 35·78 centimetres; what is the content of the block in cubic decimetres?

17. A cube contains 927·5 cubic inches; find the length of its edge.

CHAPTER XIII.

DECIMALS AND CONCRETE QUANTITIES.

Ex. 1. *Find the value of*

£4·790625.

$$\begin{array}{r}
\pounds 4{\cdot}790625 \\
20 \\
\hline
15{\cdot}812500s. \\
12 \\
\hline
9{\cdot}7500d. \\
4 \\
\hline
3{\cdot}00f.
\end{array}$$

Answer £4. 15*s.* $9\frac{3}{4}d.$

Ex. 2. *Find the value of*

3·0253 *miles.*

$$\begin{array}{rl}
3{\cdot}0253 & \\
8 & \\
\hline
{\cdot}2024 & \text{furlongs} \\
40 & \\
\hline
8{\cdot}0960 & \text{poles} \\
5\frac{1}{2} & \\
\hline
{\cdot}480 & \\
{\cdot}048 & \\
\hline
{\cdot}528 & \text{yds.} \\
3 & \\
\hline
1{\cdot}584 & \text{ft.} \\
12 & \\
\hline
7{\cdot}008 & \text{inches.}
\end{array}$$

Answer 3 m. 8 p. 1 ft. 7·008 in.

Ex. 3. *Find the value of* ·525 *of* £6. 6*s.* 8*d.*

$$\begin{array}{rl}
\pounds{\cdot}525 & \\
6\frac{1}{3} & \\
\hline
\pounds 3{\cdot}150 & = 6 \text{ times top line} \\
{\cdot}175 & = \frac{1}{3} \text{ top line} \\
\hline
\pounds 3{\cdot}325 & = 6\frac{1}{3} \text{ times top line} \\
20 & \\
\hline
6{\cdot}500s. & \\
12 & \\
\hline
6{\cdot}00d. &
\end{array}$$

Answer £3. 6*s.* 6*d.*

Here, 6*s.* 8*d.* being an easy fraction of a £, we express ·525 as a decimal of £, and multiply by $6\frac{1}{3}$.

Ex. 4. *Find the value of* ·0875 *of* £2. 18*s.* 4*d.*

·0875 of £2. 18*s.* 4*d.* = ·0875 of £2 + ·0875 of 18*s.* + ·0875 of 4*d.*

Let ·0875 = *a*.

$$\begin{array}{rl}
£\cdot0875 & = a \times £1 \\
\underline{2} & \\
£\cdot1750 & \\
\underline{20} & \\
3\cdot500s. & \\
\underline{1\cdot575s.} & = a \times 18 \text{ shillings} \\
5\cdot075s. & \\
\underline{12} & \\
\cdot900d. & \\
\underline{\cdot350d.} & = a \times 4 \text{ pence} \\
1\cdot25d. & \\
\underline{4} & \\
1\cdot00 f. &
\end{array}$$

Answer, 5*s.* $1\frac{1}{4}d.$

Or thus:

£2. 18*s.* 4*d.* = 700 pence,
·0875*d.* × 700 = 61·25 pence
= 5*s.* $1\frac{1}{4}d.$

Ex. 5. *Find the value of* $\frac{\cdot\dot{4}6153\dot{8} \times 29\cdot25}{65\cdot52 \times \cdot\dot{1}4285\dot{7}}$ *of 2 cwt. 1 qr. 8 lbs.*

$$\text{The expression} = \frac{\frac{6}{13} \times 29\frac{1}{4}}{65\frac{13}{25} \times \frac{1}{7}} \textit{ of } 9\tfrac{2}{7} \text{ qrs.}$$

$$= \frac{6 \times 117 \times 25 \times 7}{13 \times 1638 \times 4} \text{ of } \frac{65}{7} \text{ qrs.}$$

$$= \frac{3 \times 25 \times 5}{28} \text{ qrs.}$$

$$= \frac{375}{28} \text{ qrs.}$$

$$= 3 \text{ cwt. } 1 \text{ qr. } 11 \text{ lbs.}$$

It is hardly necessary to say that examples of this form do not occur in the affairs of life. In actual computation our example would assume the following form, and be solved by logarithms:

$$\frac{\cdot4615 \times 29\cdot25}{65\cdot52 \times \cdot1429} \text{ of } 2 \text{ cwt. } 1 \text{ qr. } 8 \text{ lbs.}$$

$$\text{The expression} = \frac{\cdot4615 \times 29\cdot25}{65\cdot52 \times \cdot1429} \text{ of } 2\cdot321 \text{ cwt.}$$

Let x = the value of the expression,

$$\begin{aligned} \text{then } \log x &= \log \cdot4615 + \log 29\cdot25 + \log 2\cdot321 - \log 65\cdot52 - \log \cdot1429 \\ &= \bar{1}\cdot6642 + 1\cdot4661 + \cdot3657 - 1\cdot8163 - \bar{1}\cdot1551 \\ &= \cdot5246\,; \end{aligned}$$

antilog = 3347 ;

∴ $x = 3\cdot347$ cwt.

= 3 cwt. 1 qr. 11 lb. within about $\frac{1}{10}$ of a lb.

EXAMPLES. LXII.

Find the value of :

1. £·9375 ; 1·00375 of a ton ; 1·0165 of an acre (in yds.).

2. 1·9514 of a ton + 1·5345 of a cwt. + 1·3125 of a quarter.

3. ·275 of £13. 2*s*. 6*d*. ; ·00625 of £2. 4*s*.

4. ·001 of £6. 5*s*. ; 7·5 of 3·45 of £5. 10*s*.

5. ·35 of £2. 10*s*. + ·0625 of a guinea + ·375 of 2*s*. 6*d*.

6. 2·307 of 16 cwt. 3 qrs. 13½ lbs. (to lbs.).

7. ·76 of 1 cwt. + ·275 of 1 qr. + ·555 of a lb.

8. 1·201 of a lb. – 10·056 of an oz.

9. $\dfrac{4\cdot\dot{2}8571\dot{4} \text{ of } 3\cdot4}{1\cdot3 \text{ of } 2\cdot\dot{4}2857\dot{1}}$ of $\dfrac{\cdot4\dot{3} \text{ of } \cdot625}{\cdot24}$ of 24 tons. (Prove by logarithms.)

10. 6·45 tons – 18·09375 cwts. + $3\cdot60\dot{7}1428\dot{5}$ of 168 lbs.

As we have seen, when a mixed quantity is to be expressed as a decimal of another mixed quantity, the first is expressed as a vulgar fraction of the second, and the fraction is then reduced to its decimal equivalent. When we have to express a *mixed quantity* as a decimal of a *simple unit*, we may proceed as in the examples following.

Ex. 1. *Reduce* 17*s*. 6*d*. *to the decimal of* £1.

$$\begin{array}{r|l} 12 & 6d. \\ \hline 20 & \mathbf{17}\cdot5s. \quad \text{(17s. taken down).} \\ \hline & £\cdot875. \end{array}$$

$\dfrac{6}{12} = \cdot5s.$; add 17*s*.

$\dfrac{17\cdot5}{20} = £\cdot875.$

Ex. 2. *Express* £3. 14*s.* 6*d.* *as a decimal of* £5.

12	6*d.*	
20	**14**·5*s.*	(14*s.* taken down.)
5	**£3**·725	(£3 taken down).
	£·745	

Or, by the longer method, thus :

$$\frac{£3.\ 14s.\ 6d.}{£5}=£\frac{3\frac{29}{40}}{5}$$

$$=£\tfrac{149}{200}$$

$$=£·745.$$

Ex. 3. *Express* 2 *qrs.* 21 *lbs.* 14 *oz. in decimal terms of a ton.*

	16	14 ozs.
28 {	7	**21**·875 lbs.
	4	**3**·125
	4	**2**·78125 qrs.
	20	·6953125 cwt.
		·034765625 of a ton.

Work the example by the fractional method, and note how much more labour it involves.

EXAMPLES. LXIII.

Reduce

1. 2 feet 6 inches to the decimal of a yard.
2. 3 yards 2 feet to the decimal of a mile.
3. 6 miles 5 furlongs to the decimal of 10 miles.
4. 3 roods 15 poles to the decimal of an acre.
5. 2 acres 3 roods 12 poles to the decimal of 5 acres.
6. 17 cwt. 3 qrs. 14 lbs. to the decimal of a ton.
7. 4 tons 5 cwt. 1 qr. 17½ lbs. to the decimal of a ton.
8. 8 tons 3 cwt. 2 qrs. 24½ lbs. to the decimal of 20 tons.

CHAPTER XIV.

TO DECIMALISE MONEY AT SIGHT.

THE rules devised for this purpose are based upon the following considerations:

I. Since $1s. = £\frac{1}{20} = £{\cdot}05$, we can decimalise any number of shillings by multiplying £·05 by the number of shillings, and marking off two decimal places.

Examples. 9 shillings $= £{\cdot}05 \times 9 = £{\cdot}45$;
19 shillings $= £{\cdot}05 \times 19 = £{\cdot}95$, and so on.

II. $6d. = £\frac{1}{40} = £{\cdot}025$;
$\therefore\ 1d. = \frac{1}{6}$ of $£{\cdot}025 = £{\cdot}004\frac{1}{6} = £{\cdot}001 \times 4\frac{1}{6}$.
$\frac{1}{4}d. = \frac{1}{24}$ of $6d$;
$\therefore\ \frac{1}{4}d. = \frac{1}{24}$ of $£{\cdot}025 = £{\cdot}001\frac{1}{24} = 1\frac{1}{24}$ of £·001,
$\frac{1}{2}d. = \frac{2}{24}$ of $£{\cdot}025 = £{\cdot}002\frac{2}{24} = 2\frac{2}{24}$ of £·001,
$2\frac{1}{4}d. = \frac{9}{24}$ of $£{\cdot}025 = £{\cdot}009\frac{9}{24} = 9\frac{9}{24}$ of £·001;

and so on, the resulting three-place decimal being always too small by $£{\cdot}001 \times \frac{\text{the number of farthings}}{24}$.

We can correct the error (approximately) by allowing 1 for a fraction that is ½ or more than ½.

Ex. *Express* 13*s.* 3¼*d.* *as a decimal of* £1.

Approximate Method.	*Exact Method.*
$£{\cdot}05 \times 13 = £{\cdot}65$ $3\frac{1}{4} = \frac{13}{24}$ of $£{\cdot}025 = £{\cdot}013\frac{13}{24}$ $\therefore\ 13s.\ 3\frac{1}{4}d. = £{\cdot}663\frac{13}{24}$ $= £{\cdot}664$. Error less than £·0005.	The working is the same to the third line. Decimalising (mentally) the fraction $\frac{13}{24}$, we have $541\dot{6}$. $\therefore\ £{\cdot}663\frac{13}{24} = £{\cdot}663541\dot{6}$.

Obviously, for any sum under 3*d.* the error is less than $\frac{1}{2}$ of £·001 and may, in approximation, be neglected, but for every sum of *3d. or more* allowance for the rejected fraction must be made. The rule generally observed in this approximation is:

If the number of farthings is between 11 and 36, add 1 to the third decimal place; and, if the number of farthings is 36 or more, add 2 to the third decimal place.

Ex. 1. £3. 7*s.* $2\frac{1}{4}$*d.* = £3 + 7*s.* + 9 farthings
= £3 + £·35 + £·009
= £3·359 (defect = $\frac{9}{24}$ of £·001).

Ex. 2. £3. 11*s.* $4\frac{1}{2}$*d.* = £3 + 11*s.* + 18 farthings
= £3 + £·55 + £·019
= £3·569 (excess $\frac{6}{24}$ of £·001).

Ex. 3. £2. 19*s.* $10\frac{1}{2}$*d.* = £2 + 19*s.* + 42 farthings
= £2 + £·95 + £·044
= £2·994 (excess $\frac{6}{24}$ of £·001).

The *limits of error* in the answers are £ ± ·0005, and therefore the answers are correct to three decimal places.

The same examples worked *exactly*:

(1) £3. 7*s.* $2\frac{1}{4}$*d.* = £3 + 7*s.* + 9 farthings
= £3·35 + £·009 + $\frac{9}{24}$ of £·001
= £3·359 + £($\frac{9}{24}$ of 1 in the third decimal place)
= £3·359 + £·000375
= £3·359375.

(2) £3. 11*s.* $4\frac{1}{2}$*d.* = £3. + 11*s.* + 18 farthings
= £3·55 + £·018 + $\frac{18}{24}$ of £·001
= £3·568 + £($\frac{18}{24}$ of 1 in the third decimal place)
= £3·568 + £·00075
= £3·56875.

(3) £2. 19*s*, $10\frac{1}{2}$*d.* = £2 + 19*s.* + 42 farthings
= £2·95 + £·042 + $\frac{42}{24}$ of £·001
= £2·993 + £($\frac{18}{24}$ of 1 in the third decimal place)
= £2·99375.

When, as in the last example, the remainder fraction is greater than unity, add 1 to the digit in the third place, and proceed with the reduction of the remaining fraction for the remaining places of decimals.

The following method is shorter and less troublesome.

1. To find the decimal to three places:

(*a*) Multiply the number of shillings by 5, and write the product as a two-place decimal of £1.

(*b*) Express pence and farthings as farthings (adding 1 farthing if the number of pence is 6 or over), and write the result as a three-place decimal of £1.

(*c*) Add the results of (*a*) and (*b*).

2. To find the remaining digits of the decimal:

Divide by 6 the pence and farthings or the pence and farthings diminished by 6*d.* if the pence be 6 or over.

Take 1 and 3 of the examples given above, namely, £3. 7*s.* 2¼*d.* and £2. 19*s.* 10½*d.*

Ex. 1. 7 × 5 = 35, for which write £·35
2¼*d.* = 9 farthings = £·009 approx.
sum = £·359.

$\frac{2\frac{1}{4}}{6}=\frac{2{\cdot}25}{6}={\cdot}375$, the digits to be affixed to the three-place decimal already found.

Hence £3. 7*s.* 2¼*d.* = £3·359375.

Ex. 3. 19 × 5 = 95, for which write £·95
10½*d.* + ¼*d.* = 43 farthings = £·043 approx.
sum = £·993.

$\frac{10\frac{1}{2}d. - 6d.}{6}=\frac{4{\cdot}5}{6}={\cdot}75$, the digits to be affixed to the three place decimal already found.

Hence £2. 19*s.* 10½*d.* = £2·99375.

By reversing the process we can find in shillings, etc. (within a farthing) the value of a decimal of £1. For this purpose we take the three first digits of the decimal, and proceed as shown below.

Ex. 1. £3·359|375.

5's in 35, 7, that is, 7 shillings;
4's in 9, $2\frac{1}{4}$, that is, $2\frac{1}{4}$ pence.

Answer, £3. 7*s.* $2\frac{1}{4}$*d.*

Ex. 2. £3·568|75.

5's in 56, 11 and 1 over, that is, 11 shillings and 10 farthings over (app.);
4's in 18, $4\frac{1}{2}$, that is $4\frac{1}{2}$*d.*

Answer, £3. 11*s.* $4\frac{1}{2}$*d.*

Ex. 3. £2·993|75.

5's in 99, 19 and 4 over, that is, 19 shillings and 40 farthings over (app.).
The number to be divided by 4 is 43, which would give 6 *or more*; hence we subtract 1 before dividing by 4.

4's in (43 – 1) = 4's in 42 = $10\frac{1}{2}$, that is, $10\frac{1}{2}$*d.*

Answer, £2. 19*s.* $10\frac{1}{2}$*d.*

Ex. 4. *Express to 3 decimal places £577. 12s. 6¾d. as a decimal of* £835. 2*s.* 6*d.*

£835. 2*s.* 6*d.* = £835·125,
£577. 12*s.* $6\frac{3}{4}$*d.* = £577·628125;

$$\therefore \text{ required decimal} = \frac{577\cdot628125}{835\cdot125} = \cdot692.$$

$$\begin{array}{r} \cdot6916 \\ 83{,}5{,}\cdot1{,}2{,}5\;\big)\;\overline{577\cdot628125} \\ 76\;553 \\ 1\;392 \\ 557 \\ 56 \end{array}$$

Check by logarithms:

$$\begin{aligned} \log\frac{577\cdot6}{835\cdot1} &= \log 577\cdot6 - \log 835\cdot1 \\ &= 2\cdot7617 - 2\cdot9218 \\ &= \bar{1}\cdot8399; \end{aligned}$$

antilog = 6916;

$\therefore$ *Ans.* = ·6916 = ·692, correct to 3 places.

Ex. 5. A man pays £26. 11*s.* $3\frac{3}{4}$*d.* for 17 tons 11 cwt. 3 qrs. of goods: find the cost per ton.

17 tons 11 cwt. 3 qrs. = 17·5875 tons,
£26. 11*s.* $3\frac{3}{4}$*d.* = £26·565625;

$$\therefore \text{ cost per ton} = \frac{£26\cdot565625}{17\cdot5875} = £1\cdot5104 = £1.\ 10s.\ 2\tfrac{1}{2}d.$$

$$\begin{array}{r} 1\cdot5104 \\ 17{,}\cdot5{,}8{,}7{,}5\;\big)\;\overline{26\cdot565625} \\ 8\;9781 \\ 1843 \\ 84 \\ 14 \end{array}$$

By logarithms:

Taking numerator and denominator to the nearest fourth figure, we have

$$\begin{aligned}\log\frac{26{\cdot}57}{17{\cdot}59} &= \log 26{\cdot}57 - \log 17{\cdot}59\\ &= 1{\cdot}4243 - 1{\cdot}2453\\ &= 0{\cdot}1790;\\ \text{antilog} &= 1510;\\ \therefore\ \text{cost per ton} &= £1{\cdot}510\\ &= £1.\ 10s.\ 2\tfrac{1}{2}d.\end{aligned}$$

EXAMPLES. LXIV.

Decimalise (1) to three places, and (2) exactly:

1. £2. 17*s.* 2¼*d.*; £3. 5*s.* 8½*d.*; £6. 13*s.* 9¼*d.*

2. £5. 17*s.* 1¾*d.*; £7. 3*s.* 10½*d.*; £8. 18*s.* 7¼*d.*

3. £1. 16*s.* 7½*d.*; £3. 2*s.* 6¾*d.*; £10. 17*s.* 11¼*d.*

4. £3. 19*s.* 6¼*d.*; £9. 17*s.* 8¾*d.*; £6. 12*s.* 10¼*d.*

5. Find (to the nearest farthing) the values of:

£·76935; £·06843; £·14563; £·859164.

(In the answers to 6, 7, and 8 omit fractions of pence.)

6. The cost, with carriage, of 2 tons 3 qrs. 11 lbs. of goods was £93. 12*s.* 2*d.*: what was the cost per ton?

7. The cost of repairing 6 miles 3 furlongs 18 poles of a road was £371. 13*s.* 5*d.*: find the cost per mile.

8. The cost of fencing a paddock of 26 acres 3 roods 18 perches was £50. 2*s.* 10*d.*: find the cost per acre.

CHAPTER XV.

PRACTICE.

PRACTICE is of two kinds—simple and compound. Simple practice is a short method of solving questions in compound multiplication. It is now taught at an early stage of boys' education, and we shall assume that the student is familiar with it.

Compound practice is a short method of solving questions in which we are required to find the value of a quantity of (say) goods expressed in two or more denominations, the value of a single unit of one of the higher denominations being given. The unit, the value of which is given, may be called the chief unit.

Facility in solving such questions depends largely on skill in decomposing the subordinate units into suitable aliquot parts, that is, fractional parts of a unit with unity for their numerator.

Ex. *Find the value of* 2 *tons* 3 *qrs.* 11 *lbs. of goods at* £2. 5*s.* 10*d. per cwt.*

The chief unit is cwts.

2 tons 3 qrs. 11 lbs. = 40 cwts. 3 qrs. 11 lbs.

	£.	s.	d.		
	2	5	10	= the value of 1 cwt.	
			40		
	91	13	4	= the value of 40 cwts.	
2 qrs. = $\frac{1}{2}$ of 1 cwt.	1	2	11	= ,,	2 qrs.
1 qr. = $\frac{1}{2}$ of 2 qrs.	0	11	$5\frac{1}{2}$	= ,,	1 qr.
8 lbs. = $\frac{1}{7}$ of 2 qrs.	0	3	$3\frac{2}{7}$	= ,,	8 lbs.
2 lbs. = $\frac{1}{4}$ of 8 lbs.	0	0	$9\frac{23}{28}$	= ,,	2 lbs.
1 lb. = $\frac{1}{2}$ of 2 lbs.	0	0	$4\frac{51}{56}$	= ,,	1 lb.
	£93	12	$2\frac{29}{56}$	= ,,	40 cwts. 3 qrs. 11 lbs.

Except when the answer is to be exact, this method should not be used. For accuracy to pence, work by one of the following:

tons. qrs. lbs. cwts.
2 . 3 . 11 = 40·84821 and £2. 5*s*. 10*d*. = £2·291$\dot{6}$.

4*s*. = £$\frac{1}{5}$.	£40·84821...
	2
	81·6964
1*s* = $\frac{1}{4}$ of 4*s*.	8·1696
6*d*. = $\frac{1}{2}$ of 1*s*.	2·0424
4*d*. = $\frac{1}{3}$ of 1*s*.	1·0212
	·6808
	£93·6104
	= £93. 12*s*. 2$\frac{1}{2}$*d*.

Or by multiplication, thus;

(*a*) £40·84821...
66 1922 *i.e.* 2·291$\dot{6}$ reversed.
81 6964
8 1696
3 6763
408
245
24
£93·6100
= £93. 12*s*. 2$\frac{1}{2}$*d*.

(*b*) £40·84821... × 2·291$\dot{6}$ = £22·91666... × 4·084821... .

£22·91666
28 4804
91 6666
1 8333
916
183
4
£93·6102
= £93. 12*s*. 2$\frac{1}{2}$*d*.

By logarithms:
log (£22·92 × 4·084)
= log 22·92 + log 4·084
= 1·3602 + 0·6111
= 1·9713;
antilog = 9361;
∴ *Ans*. = £93·61.

EXAMPLES. LXV.

Find (to the nearest penny) by any of the foregoing methods the values of:

1. 2 qrs. 7 ozs. 9 drs. at 18*s*. 6*d*. per lb.
2. 17 ac. 3 ro. 19 per. at £5. 18*s*. 6*d*. per acre.
3. 2 cwts. 3 qrs. 16 lbs. at £14. 7*s*. 6*d*. per cwt.
4. 27 ac. 3 ro. 27 po. at £7. 10*s*. 8*d*. per acre.
5. 2 tons 15 cwts. 27 lbs. at £5. 11*s*. 7*d*. per cwt.
6. 23 ac. 3 ro. 5 po. at 2$\frac{1}{2}$ guineas per acre.
7. 2 ac. 3 ro. 16 po. at £24. 13*s*. 4$\frac{1}{2}$*d*. per rood.

8. 17 cwts. 3 qrs. 21 lbs. at £1. 19*s*. 7½*d*. per cwt.
9. 11 cwts. 3 qrs. 23 lbs. at £4. 5*s*. 6*d*. per cwt.
10. 23 cwts. 1 qr. 18 lbs. at £19. 17*s*. 9*d*. per cwt.
11. 44 ac. 2 ro. 25 po. at £55. 16*s*. 7½*d*. per acre.
12. 24 cwts. 3 qrs. 16 lbs. 10 ozs. at £2. 0*s*. 8*d*. per cwt.
13. 10 lbs. 11 ozs. 16 dwts. 8 grs. at £3. 17*s*. 10*d*. per oz.
14. 5 lbs. 10 ozs. 16 grs. at £3. 6*s*. 9*d*. per lb.
15. 3 miles 3 fur. 120 yds. at £47. 18*s*. 6*d*. per mile.
16. 28 miles 5 fur 12 po. at £95. 18*s*. 6*d*. per mile.
17. 18 miles 6 fur. 30 po. at £7. 19*s*. 11½*d*. per mile.
18. 24 sq. yds. 1 sq. ft. 52 sq. ins. at £2. 7*s*. 3*d*. per sq. yd.
19. 26 sq. yds. 2 sq. ft. 19 sq. ins. at £6. 1*s*. 6*d*. per sq. yd.
20. 9 silver spoons each weighing 2 lbs. 11 ozs. 13 dwts. at £3. 6*s*. 11*d*. per lb.
21. 38 qrs. 3 bus. 3 pks. at £2. 13*s*. 10*d*. per qr.
22. 2 qrs. 2 lbs. 12 ozs. 13 drs. at £1. 10*s*. per lb.
23. 39 ozs. 16 dwts. 18 grs. at 9*s*. 10*d*. per oz.
24. 518 ac. 3 ro. 7¾ po. at £18. 7*s*. 6*d*. per acre.
25. 365 days 5 hrs. 48 min. at £17. 18*s*. 1*d*. per day.
26. 15 cub. yds. 13 cub. ft. 792 cub. ins. at £1. 18*s*. 3*d*. per cub. yd.
27. 17 cub. yds. 5 cub. ft. 100 cub. ins. at £2. 13*s*. 6*d*. per cub. yd.
28. 34 cub. yds. 3 cub. ft. 288 cub. ins. at £3. 6*s*. 9*d*. per cub. yd.
29. 37 cub. yds. 10 cub. ft. 624 cub. ins. at £6. 1*s*. 6*d*. per cub. yd.
30. $\cdot 8\dot{6}$ lbs. $+ 2\cdot 3\dot{2}$ dwts. $+ 2\frac{4}{5}$ grs. at £4. 4*s*. 11*d*. per oz.

CHAPTER XVI.

BANKRUPTCIES, RATING, TAXING, ETC.

A PERSON who declares or proves himself unable to pay what he owes is said to be *insolvent* or *bankrupt*. The persons to whom he owes money are called his *creditors*, and he is called their *debtor*.

The debtor's debts are called his *liabilities*, and his possessions and the money owing to him are called his *assets*.

A debtor, though unable to pay the whole of his debts, is generally able to pay a part of them, that is, he can pay a part of every pound he owes. What he can pay on every pound he owes is called the *dividend*.

Rates and taxes are moneys levied by local bodies or by the Government on the values of properties, on income, and on goods of various kinds.

Rates, dividends, and taxes on income are reckoned at so much in the pound on the value of property, on liabilities, and on income or part of income.

A rate of 4*d*. in the £ means that 4*d*. is to be paid for every £ at which the property is valued; a dividend of 3*s*. 4*d*. in the £ means that a bankrupt pays 3*s*. 4*d*. on every pound he owes; and an income-tax of 6*d*. in the £ means that the receiver of the income has to pay 6*d*. on every £ of his income or on every £ of the taxable part of his income.

A rate of 4*d*. in the £ $= \frac{4}{240}$ or $\frac{1}{60}$ of the value of the property on which it is levied.

A dividend of 3*s*. 4*d*. in the £ $= \frac{3\frac{1}{3}}{20}$ or $\frac{1}{6}$ of every pound owing, and therefore $\frac{1}{6}$ of the whole debt.

It is obvious that, if a debtor can pay only $\frac{1}{6}$ of his debt, $\frac{5}{6}$ of the debts will remain unpaid; hence his creditors will lose $\frac{5}{6}$ of what is owing to them. And generally,

$$\frac{\text{Dividend in shillings}}{20} = \text{the part of his debts a man can pay,}$$

and the complement of this fraction = the part of his debt he is unable to pay.

Ex. 1. *A bankrupt's assets are* £51, *and he can pay* 2*s.* 6*d.* *in the* £: *find* (1) *what part of his liabilities he can pay,* (2) *what part he is unable to pay,* (3) *the amount of his liabilities.*

$$2s.\ 6d. \text{ in the £.} = \frac{2\frac{1}{2}}{20} = \frac{1}{8} \text{ of his liabilities;}$$

$$\therefore \text{ he is unable to pay } \frac{7}{8} \text{ of his liabilities.}$$

$$\text{He pays } \frac{1}{8} \text{ of his liabilities; the amount he pays is £51;}$$

$$\therefore\ \frac{1}{8} \text{ of his liabilities} = £51;$$

$$\therefore \text{ his liabilities} = £51 \times 8 = £408.$$

Again, if a man has £3 and owes £5, it is obvious that he can pay only $\frac{3}{5}$ of his debt, and that therefore $\frac{2}{5}$ of his debt will remain unpaid. If he owes £150 and has only £65, he can pay only $\frac{65}{150}$ of his debt; and the part of his debt he is unable to pay may be represented by the complement of this fraction, that is, $\frac{85}{150}$ of his debts.

And, generally,

$\frac{\text{assets}}{\text{liabilities}}$ = the part of his liabilities a man can pay, and also the part of every pound of his liabilities he can pay.

Ex. 2. *A bankrupt's debts are* £372, *and his assets* £34: *find* (1) *what dividend in the* £ *he can pay, and* (2) *what part of his debts he cannot pay.*

He can pay $\frac{34}{272}$ or $\frac{1}{8}$ of his liabilities, and therefore $\frac{1}{8}$ of every pound he owes; $\therefore$ dividend = 2*s.* 6*d.* in the £.

He can pay $\frac{1}{8}$ of his liabilities;

$$\therefore\ \tfrac{7}{8} \text{ of his liabilities} = \text{the part of them he cannot pay.}$$

In the actual affairs of life, the relation of assets to liabilities is often such that, to obtain the dividend without undue labour, we should use the method of approximation as in the following.

Ex. 3. *A bankrupt's assets are £577. 12s. 6¾d. and his liabilities £835. 2s. 6d. : find, to the nearest penny, what dividend in the £ he can pay.*

$$\text{Assets} = £577.\ 12s.\ 6\tfrac{3}{4}d. = £577{\cdot}628125,$$

$$\text{liabilities} = £835.\ 2s.\ \ 6d. \ = £835{\cdot}125\,;$$

$$\therefore \text{ dividend in the } £ = \frac{£577{\cdot}628125}{£835{\cdot}125}$$

$$= £{\cdot}6916 \ldots \text{(by contracted division)}$$

$$= £{\cdot}692 = 13s.\ 10d.$$

Work the example by means of logarithms.

Ex. 4. *A bankrupt can pay 14s. 9d. in the £; his liabilities amount to £13901. 6s. 8d. : find the amount of his assets.*

$$\text{His assets} = \frac{14\frac{3}{4}}{20} \text{ of his liabilities}$$

$$= \frac{14\frac{3}{4}}{20} \text{ of } £13901.\ 6s.\ 8d.$$

The following method is often more expeditious.

	£13901·33333...
$10s. = £\frac{1}{2}$	6950·66666...
$4s. = £\frac{1}{5}$	2780·26666...
$6d. = \frac{1}{8}$ of $4s.$	347·53333...
$3d. = \frac{1}{2}$ of $6d.$	173·76666...
	£10252·2333 ...

Assets = £10252. 4s. 8d.

Harder examples :

Ex. 5. *Find a man's gross rental, if after paying an income-tax of 6d. in the pound on the whole, and 3s. 6d. in the pound on $\frac{3}{4}$ of his rental, his net income is £2700.*

He first pays 6d. in the pound on the whole rental ;

$$\therefore \text{ he pays } \tfrac{6}{240}, \text{ or } \tfrac{1}{40} \text{ of his rental;}$$

he then pays 3s. 6d. in the pound on $\frac{3}{4}$ of his rental ;

$$\therefore \text{ he pays } \frac{3\frac{1}{2}}{20} \text{ of } \tfrac{3}{4}, \text{ or } \tfrac{21}{160} \text{ of his rental;}$$

$$\therefore \text{ total amount paid in taxes} = \left(\tfrac{1}{40} + \tfrac{21}{160}\right) \text{ of his rental}$$

$$= \tfrac{5}{32} \text{ of his rental;}$$

$$\therefore \text{ he has } \tfrac{27}{32} \text{ of his rental left;}$$

but he has £2700 left ;

$$\therefore \tfrac{27}{32} \text{ of gross rental} = £2700\,;$$

$$\therefore \text{ gross rental} = £\tfrac{2700 \times 32}{27} = £3200.$$

The student should work the example, putting x for the gross rental.

Ex. 6. *A creditor receives on a debt of* £296 *a dividend of* 12*s.* 4*d. in the pound, and he receives a further dividend of* 3*s.* 9*d. in the pound upon the deficiency: find how much the creditor receives in all.*

$$\text{1st payment} = \frac{12\frac{1}{3}}{20} \text{ of the debt}$$

$$= \tfrac{37}{60} \text{ of the debt;}$$

$$\therefore\ \tfrac{23}{60} \text{ of the debt remains unpaid;}$$

$$\therefore\ \text{the deficiency} = \tfrac{23}{60} \text{ of the debt.}$$

$$\text{2nd payment} = \frac{3\frac{3}{4}}{20} \text{ of the deficiency,}$$

$$\text{or } \frac{3\frac{3}{4}}{20} \text{ of } \tfrac{23}{60} \text{ of the debt} = \tfrac{23}{320} \text{ of the whole debt;}$$

$$\therefore\ \text{1st payment} + \text{2nd payment} = (\tfrac{37}{60} + \tfrac{23}{320}) \text{ of the whole debt;}$$

$$\therefore\ \text{the creditor receives } \tfrac{661}{960} \text{ of the whole debt;}$$

$$\therefore\ \text{he receives } \tfrac{661}{960} \text{ of £296, or £203. 16}s.\ 2d.$$

Work the example, putting x for the amount of the debt.

Ex. 7. *When the income-tax is* 7*d. in the pound a person has to pay* £63 *less than when the tax was* 11*d. in the pound: find his income.*

The first tax was $\frac{11}{240}$ of his income;

the reduced tax is $\frac{7}{240}$ of his income;

and the difference between these amounts is £63;

$$\therefore\ (\tfrac{11}{240} - \tfrac{7}{240}) \text{ of his income} = £63;$$

$$\therefore\ \tfrac{1}{60} \text{ of his income} = £63;$$

$$\therefore\ \text{income} = £63 \times 60 = £3780.$$

Work the example, putting x for the amount of the income.

EXAMPLES. LXVI.

1. A bankrupt's debts amounted to £980, and his creditors received 13*s.* 6*d.* in the £: find the value of his effects.

2. How much will a poor-rate of 2*s.* 8*d.* in the £ produce in a parish in which the whole property is rated at £4736. 5*s.*?

3. A man's debts amount to £5069. 10*s.*, and he can pay 14*s.* 11½*d.* in the £: find the value of his assets.

4. If a tax of 6*d.* in the £ be levied upon the excess of all incomes over and above the first £200, what tax will a gentleman have to pay whose annual income is 500 guineas?

5. A bankrupt pays 3*s.* 6½*d.* in the pound, and his assets amount to £666. 16*s.* 1¾*d.*: find the amount of his debts.

6. *A* pays an income-tax of £63. 14*s.* 5¼*d.* at the rate of 7*d.* in the £: find his income.

7. A creditor received 16*s.* 3*d.* in the pound, and thereby lost £135. 10*s.*: how much was due to him?

8. The poor-rate being 3*s.* 4*d.* in the £, what is the rateable value per annum of a property which pays annually a rate of £416. 13*s.* 4*d.*?

9. A rate of 2*s.* 9*d.* in the £ on the rental of a parish produces £352: what is the rateable value of the parish?

10. A bankrupt's debts amount to £3500, and his assets to £1370. 16*s.* 8*d.*: find how much his estate will pay in the £.

11. If a bankrupt whose effects are worth £1181. 12*s.* 6*d.*, owes £4726. 10*s.*, how much will he be able to pay in the £?

12. A bankrupt's debts amount to £5342. 5*s.*, and his creditors lose £1736. 4*s.* 7½*d.*: find how much in the £ the bankrupt pays.

13. A bankrupt's debts amount to £15490, and his assets to £5615. 2*s.* 6*d.*: how much will a creditor receive on £237. 10*s.*?

14. If a bankrupt's debts amount to £7250, and his creditors lose £6011. 9*s.* 2*d.*: how much in the £ does his estate pay?

15. A bankrupt owes to one creditor 1000 guineas, to each of two others £500, and to each of three others £150: his assets are £600: find how much he can pay in the £, and how much the first creditor will receive.

16. After paying an income-tax of 8*d.* in the £, a man has a net income of £759. 18*s.* 5*d.*: find his gross or nominal income.

17. A man has a net income of £1755 after paying an income-tax of 6*d.* in the £: find his gross income.

18. Find the gross rental of a man who, after paying an income-tax of 7*d.* in the £, and a poor-rate of 3*d.* in the £, has £307. 18*s.* 3*d.* left.

19. Find a man's gross rental if, after paying an income-tax of 8*d.* in the £, and 2*s.* 6*d.* in the £ on two-thirds of his rental, he has a net income of £398. 16*s.* 6*d.*

20. The cost of collecting a rent-charge was 6*d.* in the £ on the net receipts: the rates were 4*s.* in the £ on the gross rent-charge, and an income-tax of 4*d.* in the £ was paid on the remainder: those charges deducted, the owner received £383. 10*s.*: find the gross rent-charge, the amount of the rates, and of the income-tax, and also the cost of collecting.

21. A bankrupt owes three creditors, *A*, *B*, and *C*, £290, £556, and £620 respectively: his assets are worth £412. 6*s.* 3*d.*: what ought each creditor to receive?

22. A bankrupt's debts amount to £1700, and his assets to £900. 15*s.*: after paying costs his creditors receive 5*s.* 9*d.* in the £: find the amount of the costs.

23. A gentleman left by will to *A* £1000, to *B* £950, to *C* £800, and to *D* £750: his estate produced only £2900: how much did each legatee receive?

CHAPTER XVII.

RATIO, PROPORTION, VARIATION.

We can compare two quantities of the same kind in two ways, (1) by finding how much one of them is greater than the other, which is the *method of difference*, and (2) by finding what multiple, part, or parts one is of the other, which is called the *method of ratio.*

We have already seen that ratio is the relation that one quantity bears to another in respect of the number of times the former contains the latter, and that the measure of this relation is denoted by a fraction that has for its numerator the number of units in the first quantity and for denominator the number of units of the same kind in the second quantity.

If a and b be quantities of the same kind and expressed in terms of the same unit, then, as we have seen, the ratio of a to b is denoted by the fraction $\frac{a}{b}$ or by the form $a:b$, which means a divided by b; and the ratio of b to a is denoted by $\frac{b}{a}$ or $b:a$.

In $\frac{a}{b}$, a and b are called the terms of the ratio, the numerator a being called the *antecedent* and the denominator b the *consequent.*

If, in the ratio $\frac{a}{b}$, a is equal to b, the ratio is called a *ratio of equality*; if a is less than b, the ratio is called a *ratio of less inequality*; and, if a is greater than b, the ratio is called a *ratio of greater inequality.* That is:

A ratio is called a *ratio of equality*, *of less inequality*, or *of greater inequality*, according as its antecedent is equal to, less than, or greater than, its consequent.

The ratio of a to b is $\frac{a}{b}$; hence, since multiplying or dividing the terms of a fraction by the same number does not alter the value of the fraction,

$$\text{the ratio } \frac{a}{b}=\frac{3a}{3b}=\frac{\frac{1}{3}\text{ of }a}{\frac{1}{3}\text{ of }b}, \text{ and so on.}$$

That is, *the antecedent and the consequent may be both multiplied or both divided by the same number without alteration in the value of the ratio.*

When two or more ratios are multiplied together, they are said to be *compounded.* Thus, the ratio compounded of the ratios $2:3$ and $5:6$

$$=\tfrac{2}{3}\times\tfrac{5}{6}=\tfrac{10}{18}=\tfrac{5}{9}=5:9;$$

and the ratio compounded of the ratios $a:b$ and $c:d$

$$=\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}=ac:bd.$$

A ratio multiplied by itself is called the *duplicate* of the ratio. Thus, the ratio $a:b$ compounded with itself

$$=\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2}=a^2:b^2,$$

which is the duplicate ratio of $a:b$.

$$\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}=\frac{a^3}{b^3}=a^3:b^3,$$

which is called the triplicate ratio of $a:b$.

EXAMPLES. LXVII.

Compound the following ratios:

1. $2:5$, $4:3$, $5:4$, $8:9$.

2. $a:b$, $c:d$, $e:f$, $d:c$.

3. $ab:bc$, $bc:cd$, $bc:ab$.

4. Write down the duplicate and the triplicate of each of the following ratios: $2:3$, $5:7$, $c:d$, $xy:z$, $ab:cd$.

Compare the ratio of 4 *sheep to* 8 *sheep with the ratio of* 44 *horses to* 88 *horses.*

The first ratio is $\frac{4}{8}$, or $\frac{1}{2}$.
The second ratio is $\frac{44}{88}$, or $\frac{1}{2}$.

Observe here that the first term of the first ratio has the same relation to the second term that the first term of the second ratio has to the second term of the second ratio, that is, $\frac{1}{2}$. Hence it is said that

4 sheep are to 8 sheep as 44 horses are to 88 horses.

This may be expressed

4 sheep : 8 sheep : : 44 horses : 88 horses,
or 4 sheep : 8 sheep = 44 horses : 88 horses,

the sign : having the same meaning as the sign ÷, and the sign : : the same meaning as the sign =.

Hence $\frac{4}{8}=\frac{44}{88}$; and, since these fractions are equal, the ratios are equal; but the terms of equal ratios are said to be proportional, or to be in proportion. Hence the terms 4, 8, 44, 88 are proportionals.

Proportion is the equality of ratios; and, in every proportion, the fractions representing the ratios are identical when reduced to their lowest terms. We have seen that our example answers to this test.

Generally, if a, b, c, d are proportionals, then

$$\frac{a}{b}=\frac{c}{d}, \text{ that is, } a:b=c:d.$$

The first and last terms of a proportion are called the *extremes*, and the second and third terms the *means*; and, *in every proportion, the product of the extremes is equal to the product of the means.*

General proof:	Particular proof:
$\frac{a}{b}=\frac{c}{d}$;	$\frac{4}{8}=\frac{44}{88}$;
$\therefore\ ad=bc.$	$\therefore\ 4\times 88=44\times 8.$

From this it follows that, if the product of two numbers is equal to the product of two other numbers, the four factors can be arranged in a proportion.

The arrangement of the terms in such derived proportion must be in accordance with the conditions previously enunciated, namely, that in every proportion.

(*a*) the product of the extremes equals the product of the means,

and (*b*) the fractions denoting the ratios are identical when reduced to their lowest terms.

Consider the following proportion and its derivatives:

$$16 : 4 = 12 : 3,$$

for $16 \times 3 = 4 \times 12$, and $\frac{16}{4} = \frac{4}{1}$, and $\frac{12}{3} = \frac{4}{1}$;

(derived) $4 : 16 = 3 : 12$,

for $4 \times 12 = 16 \times 3$, and $\frac{4}{16} = \frac{1}{4}$, and $\frac{3}{12} = \frac{1}{4}$;

(derived) $16 : 12 = 4 : 3$,

for $16 \times 3 = 12 \times 4$, and $\frac{16}{12} = \frac{4}{3}$, and $\frac{4}{3} = \frac{4}{3}$;

(derived) $3 : 4 = 12 : 16$,

for $3 \times 16 = 4 \times 12$, and $\frac{3}{4} = \frac{12}{16}$, that is, $\frac{3}{4}$.

The General Case.

If a, b, c, d are proportionals, then

$$\frac{a}{b} = \frac{c}{d}, \text{ that is, } a : b = c : d;$$

$$\therefore\ ad = bc.$$

Proportions derived from this equation.

(1) Divide each side of the equation by cd, then

$$\frac{ad}{cd} = \frac{bc}{cd}; \text{ and, cancelling common factors,}$$

we have $\frac{a}{c} = \frac{b}{d}$, that is, $a : c = b : d$.

(2) Divide each side of the equation by ab, then

$$\frac{ad}{ab} = \frac{bc}{ab}; \text{ and, cancelling common factors,}$$

we have $\frac{d}{b} = \frac{c}{a}$, that is $d : b = c : a$.

(3) Divide each side of the equation by ac, then

$$\frac{ad}{ac} = \frac{bc}{ac}; \text{ and, cancelling common factors,}$$

we have $\frac{d}{c} = \frac{b}{a}$, that is, $d : c = b : a$.

If any one of the foregoing proportions is given, the three others follow; for they all express the fundamental relation that *the product of the extremes is equal to the product of the means.*

By applying the theorem that *the product of the extremes is equal to the product of the means*, we can find any term of a proportion when the three others are known.

In the following proportions the unknown term is denoted by x.

(1)
$$x : 6 = 7 : 14;$$
$$\therefore\ 14x = 42;\ \therefore\ x = \tfrac{42}{14} = 3.$$
$$\text{Proof: } 3 \times 14 = 6 \times 7 \text{ and } \tfrac{3}{6} = \tfrac{7}{14}.$$

(2)
$$5 : x = 9 : 27;$$
$$\therefore\ 9x = 27 \times 5;\ \therefore\ x = \tfrac{27 \times 5}{9} = 15,$$
$$\text{Proof: } 5 \times 27 = 15 \times 9, \text{ and } \tfrac{5}{15} = \tfrac{9}{27}.$$

(3)
$$8 : 4 = x : 12;$$
$$\therefore\ 4x = 12 \times 8;\ \therefore\ x = \tfrac{12 \times 8}{4} = 24.$$
$$\text{Proof: } 8 \times 12 = 4 \times 24, \text{ and } \tfrac{8}{4} = \tfrac{24}{12}.$$

(4)
$$18 : 6 = 6 : x;$$
$$\therefore\ 18x = 36;\ \therefore\ x = \tfrac{36}{18} = 2.$$
$$\text{Proof: } 18 \times 2 = 6 \times 6, \text{ and } \tfrac{18}{6} = \tfrac{6}{2}.$$

Three numbers, or three quantities of the same kind, are said to be in *continued proportion* when the ratio of the first to the second is equal to the ratio of the second to the third.

Thus, 2, 4, 8 are in *continued proportion* because

$$2 : 4 = 4 : 8.$$

A continued proportion is therefore a proportion in which the means are equal.

The general case:

If a, b, c are in continued proportion, then

$$a : b = b : c;$$
$$\therefore\ ac = b^2, \text{ and } \therefore\ b = \sqrt{ac}.$$

The quantity b is called a *mean proportional* between a and c.

Hence to find a mean proportional to two given numbers or quantities, multiply the two numbers together, and take the square root of the product.

The quantity c in the continued proportion a, b, c is called a *third proportional* to a and b.

Ex. 1. *Find the mean proportional to* 3 *and* 27.

Let x represent the mean proportional,

$$\text{then} \quad 3 : x = x : 27;$$

$$\therefore \; x^2 = 27 \times 3 = 81;$$

$$\therefore \; x = \sqrt{81} = 9.$$

$$\text{Proof:} \quad 3 \times 27 = 9 \times 9, \text{ and } \tfrac{3}{9} = \tfrac{9}{27}.$$

Ex. 2. *Find the third proportional to* 4 *and* 10.

Let x represent the third proportional,

$$\text{then} \quad 4 : 10 = 10 : x;$$

$$\therefore \; 4x = 10 \times 10;$$

$$\therefore \; x = \frac{10 \times 10}{4} = 25.$$

$$\text{Proof:} \quad 4 \times 25 = 10 \times 10, \text{ and } \tfrac{4}{10} = \tfrac{10}{25}.$$

The student should note carefully:

(*a*) that it is only in connection with continued proportion that we have to do with a third proportional and a mean proportional;

(*b*) that, though in a continued proportion there are four terms, there are only three *different* terms; and

(*c*) that in continued proportion the consequent of the first ratio is the antecedent of the second.

The following theorem is of great importance:

If the consequent of the preceding ratio be the antecedent of the succeeding ratio, and any number of such ratios be taken, the ratio that arises from their composition is the ratio of the first antecedent to the last consequent.

Let $a : b$, $b : c$, $c : d$ be the ratios,
then, compounding the ratios, we have

$$\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d}.$$

Cancel the common factors of the numerator and denominator, that is, divide each by bc, then

$$\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{d} = a : d.$$

Ex. 1. *Given that* $a:b=3:4$, $b:c=5:6$, $c:d=7:8$, *to find the ratio of* a *to* d.

$$\frac{a}{b}=\frac{3}{4},\ \frac{b}{c}=\frac{5}{6},\ \text{and}\ \frac{c}{d}=\frac{7}{8};$$

$$\therefore\ \frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}=\frac{3}{4}\times\frac{5}{6}\times\frac{7}{8};$$

$$\therefore\ \frac{a}{d}=\frac{35}{64},\ \text{that is, } a:d=35:64.$$

$$\text{Hence}\quad a=\frac{35}{64}\text{ of } d,\ \text{and } d=\frac{64}{35}\text{ of } a.$$

Ex. 2. *Given that* $a:b=4:5$, $c:b=6:7$, $d:c=8:9$, *if* d *is* £40, *what is* a*?*

We have already learnt that, if four quantities of the same kind are proportionals, they will be proportionals by inversion, that is, the second will be to the first as the fourth to the third.

Hence we may arrange the second and third of our proportions as follows:

$$b:c=7:6,\ \text{and}\ c:d=9:8.$$

The proportions are now $\frac{a}{b}=\frac{4}{5}$, $\frac{b}{c}=\frac{7}{6}$, and $\frac{c}{d}=\frac{9}{8}$.

Compounding the ratios, we have

$$\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}=\frac{4}{5}\times\frac{7}{6}\times\frac{9}{8};$$

$$\therefore\ \frac{a}{d}=\frac{21}{20};$$

$$\begin{aligned}\text{that is,}\quad a&=\frac{21}{20}\text{ of } d\\ &=\frac{21}{20}\text{ of £40}\\ &=\text{£42}.\end{aligned}$$

Ex. 3. *In a* 100 *yards' race,* A *can beat* B *by* 15 *yards, and* B *can beat* C *by* 5 *yards; by how much could* A *beat* C *in* 100 *yards?*

$$A\text{'s rate} : B\text{'s}=100:85,$$

$$B\text{'s rate} : C\text{'s}=100:95;$$

$$\therefore\ \frac{A\text{'s rate}}{B\text{'s rate}}\times\frac{B\text{'s rate}}{C\text{'s rate}}=\frac{100}{85}\times\frac{100}{95}=\frac{400}{323};$$

$$\therefore\ \frac{A\text{'s rate}}{C\text{'s rate}}=\frac{400}{323},$$

that is, in the same time, whatever the length of the race, C runs $\frac{323}{400}$ of the distance run by A;

$\therefore$ while A runs 100 yards, C runs $\frac{323}{400}$ of 100 yards or $80\frac{3}{4}$ yards.

$\therefore$ A can beat C by $19\frac{1}{4}$ yards.

EXAMPLES. LXVIII.

Find the values of x in

1. $7 : 8 = x : 168$.

2. $9 : x = 6 : 108$.

3. $x : 10 = 21 : 14$.

4. $17 : 51 = 6 : x$.

5. $p : q = x : y$.

6. $y : x = m : n$.

7. $m : n = x : y$.

8. $x : 4 = y : 3$.

9. Form three proportions from the proportion

$$a : b = 5 : 9.$$

Find the mean proportional to

10. 16 and 4.

11. 9 and 16.

12. 4 and 49.

13. x and y.

14. 32 acres and 8 acres.

15. a and $4b$.

Find the third proportional to

16. 5 and 10.

17. 9 and 21.

18. 25 and 15.

19. $\frac{2}{3}$ and $\frac{1}{4}$.

20. $\frac{1}{2}$ and $\frac{5}{6}$.

21. a and b.

22. If $a : b = 2 : 3$, $b : c = 3 : 4$, $c : d = 5 : 6$, find the ratio of a to d.

23. If $a : b = 4 : 5$, $c : b = 6 : 7$, and $d : c = 10 : 8$, find the value of a if d is 225.

24. If $b : a = 7 : 5$, $b : c = 9 : 14$, $d : c = 14 : 15$, and $e : d = 9 : 7$, and the value of e is £784, find the value of b.

25. A can beat B by 12 yards in a race of 150 yards, and B can beat C by 10 yards in 150 yards: by how much could A beat C in 150 yards?

26. In a quarter-mile race A can beat B by 80 yards, and B can beat C by 40 yards: by how much could A beat C in the same distance?

27. At a game of skill A can give B 5 points out of 30, and C 3 points out of 20: how many points could C give B out of 153?

28. A can beat B by 80 yards in a half-mile race, and C by 25 yards in a quarter-mile race: by how much could C beat B in a race of 498 yards?

29. A can give B 10 yards in 200 yards, B can give C 5 yards in 150, and C can give D 3 yards in 135: what handicap can A give D in a race of 300 yards?

30. A can give c yards in d yards to B, and x yards in y yards to C: what is the ratio of B's speed to C's?

Other things being constant, if 1 lb. of tea costs 2*s*.

2 lbs. of the same tea will cost 2 times 2*s*.,
3 lbs. will cost 3 times 2*s*.,
4 lbs. will cost 4 times 2*s*.; and so on,

the number of lbs. of tea and the total cost being so related that, when the former changes, the latter changes *in the same ratio.*

The mathematical expression for "changing in the same ratio" is "varying directly as" or simply "varying as," or "directly proportional to" or simply "proportional to."

Hence we may express the above relations by saying that

> the cost of any quantity of tea *varies as* or is *proportional to* the number of lbs. purchased.

The terms that change are the number of lbs. and the total cost; these terms are therefore called *variables.*

The cost per lb. remains unchanged, and is therefore called a *constant.*

If we denote the total cost by a, the constant by m, and the number of lbs. by b, we have

$$a = mb,$$

from which we derive the following equations:

$$\frac{a}{b} = m, \text{ and } \frac{a}{m} = b.$$

And, generally, if a varies as b, the fraction $\frac{a}{b}$ expresses a constant ratio.

Illustrations of *direct variation* or *direct proportion*:

(1) The total cost of any quantity of goods *varies as*, or is *proportional to*, the price per unit of the goods.

(2) The work done in any given time *varies as*, or is *proportional to*, the number of workers of the same working capacity.

(3) The time taken to do a piece of work *varies as*, or is *proportional to*, the quantity of work.

(4) The time taken to travel at a certain rate from one place to another *varies as*, or is *proportional to*, the distance.

Ex. 1. *If* 10 *lbs. of tea cost* 15*s., what should be paid for* 30 *lbs. of the same tea?*

Here 2, 3, 4 ... times the quantity would give 2, 3, 4 ... times the total cost; hence the total cost is proportional to the quantity, that is, the ratio of the total costs is equal to the ratio of the quantities.

$$\therefore \text{ cost} = 15s. \times \frac{30}{10} = £2.\ 5s.\ 0d.$$

Or let x represent the required cost in shillings,

$$\text{then}\quad x\text{s} : 15\text{s.} = 30 \text{ lbs.} : 10 \text{ lbs.};$$

$$\therefore\ \frac{x}{15} = \frac{30}{10};$$

$$\therefore\ x = \frac{30 \times 15}{10}\text{s.} = 45\text{s.} = £2.\ 5\text{s.}$$

Such questions can, of course, be solved by what is called the Unitary Method. Thus:

$$\text{The cost of } 10 \text{ lbs.} = 15\text{s.};$$

$$\therefore\ \text{cost of } 1 \text{ lb.} = \frac{15}{10}\text{s.};$$

$$\therefore\ \text{cost of } 30 \text{ lbs.} = \frac{15 \times 30}{10}\text{s.} = £2.\ 5\text{s.}$$

Ex. 2. *If* 9 *cwt. of goods cost* £18. 16*s.* 8*d.*, *what would* 2 *cwt.* 3 *qrs.* 7 *lbs. of the same goods cost at the same rate?*

In this case $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ ... of the goods would cost $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ ... of the money; hence the cost is proportional to the quantities, that is, the ratio of the two costs is equal to the ratio of the two quantities of goods.

2 cwt. 3 qrs. 7 lbs. = 45 half-stones, and 9 cwt. = 144 half-stones;

$$\therefore\ \text{cost} = £18.\ 16\text{s.}\ 8d. \times \frac{45}{144}$$

$$= £5.\ 17\text{s.}\ 8\tfrac{1}{2}d.$$

Or thus, x representing the cost in £. *s.* *d.*:

$$\frac{x}{£18.\ 16\text{s.}\ 8d.} = \frac{45}{144} = \frac{5}{16};$$

$$\therefore\ x = £18.\ 16\text{s.}\ 8d. \times \frac{5}{16}, \text{ and so on.}$$

Or thus, by the Unitary Method:

$$144 \text{ half-stones cost } £18.\ 16\text{s.}\ 8d.;$$

$$\therefore\ 1 \text{ half-stone cost } \frac{£18.\ 16\text{s.}\ 8d.}{144};$$

$$\therefore\ 45 \text{ half-stones cost } \frac{£18.\ 16\text{s.}\ 8d. \times 45}{144}, \text{ and so on.}$$

In the solutions of Exs. 1 and 2, by proportion, the student should note:

(1) that, when the required term of one ratio is greater than the corresponding term of the other, both ratios are improper fractions, that is, they are both ratios of greater inequality; and

(2) that, when the required term of one of the ratios is less than the corresponding term of the other, both ratios are proper fractions, that is, they are both ratios of less inequality.

Is the answer sought greater or less than the given term of the incomplete ratio? is the question that determines the kind of ratios that make the proportion.

Consider the case of trains travelling the same distance at different rates.

Length of journey.	Rate per hour.	Time.
A, 80 miles,	10 miles,	8 hours.
B, 80 miles,	(10×2) miles,	$\frac{1}{2}$ of 8 hours or 4 hours.
C, 80 miles,	(10×4) miles,	$\frac{1}{4}$ of 8 hours or 2 hours.
D, 80 miles,	(10×8) miles,	$\frac{1}{8}$ of 8 hours or 1 hour.

Here 2 times the rate gives $\frac{1}{2}$ of the time,
4 times the rate gives $\frac{1}{4}$ of the time,
and 8 times the rate gives $\frac{1}{8}$ of the time.

The time therefore varies, not as the rate, but as the *reciprocal* of the rate.

In each case the variables are the time and the rate, and the constant is the length of the journey.

Note that in each case rate × time = 80, that is, the product of the numbers denoting the variables is equal to the number denoting the length of the journey.

Let a denote the rate, b the time, and m the constant,

$$\text{then} \quad ab = m;$$

$$\therefore \quad a = \frac{m}{b} = m \times \frac{1}{b},$$

$$\text{and} \quad b = \frac{m}{a} = m \times \frac{1}{a};$$

$$\therefore \quad a \text{ varies as } \frac{1}{b}, \text{ and } b \text{ varies as } \frac{1}{a}.$$

Hence we have two variables, a and b, so related

(1) that, as the values of a and b change, the product ab has always the same value, and

(2) that the first varies as the reciprocal of the second, and the second as the reciprocal of the first.

Quantities having these relations are said to *vary inversely* or to be *inversely proportional.*

Illustrations of *inverse variation* or *inverse ratio*:

(1) The time taken to travel a certain distance *varies inversely as*, or is *inversely proportional to*, the rate of travelling.

(2) The number of lbs. of any material that can be purchased for a certain sum of money *varies inversely as*, or is *inversely proportional to*, the price per lb. of the material.

(3) The time taken to do a certain piece of work *varies inversely as*, or is *inversely proportional to*, the number of workers.

(4) The weight of material that a carrier will carry for a certain sum of money *varies inversely as*, or is *inversely proportional to*, the distance the weight has to be carried.

Ex. 1. *If* 12 *men can reap a field in* 4 *days, in how many days of the same length can* 32 *men of the same working capacity reap it?*

The time is inversely proportional to the number of workers, that is, 2, 3, 4 ... times the number of workers would take $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ ... of the time.

Hence 32 workers will take $\frac{12}{32}$ of the time taken by 12 workers;

$$\therefore \text{ time} = \tfrac{12}{32} \text{ of 4 days} = 1\tfrac{1}{2} \text{ days.}$$

Or thus by proportion:

Let x represent the required number of days,

$$\text{then } x \text{ days} : 4 \text{ days} = 12 \text{ men} : 32 \text{ men};$$

$$\therefore \frac{x}{4} = \frac{12}{32};$$

$$\therefore x = \frac{12 \times 4}{32} \text{ days} = 1\tfrac{1}{2} \text{ days.}$$

Ex. 2. *If* 7 *oxen or* 11 *horses eat a certain quantity of food in* 37 *days, in how many days will* 5 *oxen and* 8 *horses eat it?*

In eating capacity

$$7 \text{ oxen} = 11 \text{ horses}; \quad \therefore 1 \text{ ox} = \tfrac{11}{7} \text{ horses};$$

$$\therefore 5 \text{ oxen} = (\tfrac{11}{7} \times 5) \text{ horses} = 7\tfrac{6}{7} \text{ horses};$$

$$\therefore 5 \text{ oxen} + 8 \text{ horses} = (7\tfrac{6}{7} + 8) \text{ horses} = 15\tfrac{6}{7} \text{ horses.}$$

The time is inversely proportional to the number of horses, that is, 2, 3, 4, ... times the number of horses would take $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ ... of the time;

$$\therefore \text{ time} = \frac{11}{15\frac{6}{7}} \text{ of 37 days} = 25\tfrac{2}{3} \text{ days.}$$

Or thus by proportion:

Let x represent the required number of days,

$$\text{then } x \text{ days} : 37 \text{ days} = 11 \text{ horses} : 15\tfrac{6}{7} \text{ horses};$$

$$\therefore \frac{x}{37} = \frac{11}{15\frac{6}{7}};$$

$$\therefore x = \frac{11 \times 37}{15\frac{6}{7}} \text{ days} = 25\tfrac{2}{3} \text{ days.}$$

Ex. 3. *The temperature remaining the same the space occupied by a gas varies inversely as the pressure; and at a constant temperature a mass of air occupies 25 cubic feet under a pressure of 10 lbs. to the square inch: find what space it would occupy under a pressure of 25 lbs. to the square inch.*

The variation being inverse, greater pressure gives less space, less in the ratio $\frac{10}{25}$;

$$\therefore \text{ space} = 25 \text{ cubic feet} \times \tfrac{10}{25}$$
$$= 10 \text{ cubic feet.}$$

Or thus by proportion:

Let x represent the space in cubic feet,

then $$\frac{x}{25 \text{ cub. ft.}} = \frac{10}{25};$$
$$\therefore x = 25 \text{ cub. ft.} \times \tfrac{10}{25}.$$

Ex. 4. *Given that the area of a circle varies as the square of its radius, and that the area of a circle is 154 sq. yards when the radius is 7 yards, find the area of a circle the radius of which is $10\frac{1}{2}$ yards.*

The variation being direct, the greater radius gives the greater area, greater in the ratio $\dfrac{(10\frac{1}{2})^2}{7^2}$;

$$\therefore \text{ area} = 154 \text{ sq. ft.} \times \frac{(10\frac{1}{2})^2}{7^2}$$
$$= 154 \text{ sq. yds.} \times \frac{21 \times 21}{2 \times 2 \times 7 \times 7}$$
$$= 346\tfrac{1}{2} \text{ sq. yds.}$$

Also, if x is put for the area, we have the equation:

$$\frac{x}{154 \text{ sq. yds.}} = \frac{(10\frac{1}{2})^2}{7^2};$$
$$\therefore x = 154 \text{ sq. yds.} \times \frac{(10\frac{1}{2})^2}{7^2}.$$

The working of examples in proportion may often be greatly shortened by the use of logarithms. We work one example by means of them.

Ex. 5. *If 17·38 tons of material cost £87. 17s., what should be paid for 8·375 tons of it?*

Let £x = the cost of 8·375 tons,

then $$x = \frac{£87{\cdot}85 \times 8{\cdot}375}{17{\cdot}38};$$
$$\therefore \log x = \log 87{\cdot}85 + \log 8{\cdot}375 - \log 17{\cdot}38$$
$$= 1{\cdot}9437 + {\cdot}9230 - 1{\cdot}2400$$
$$= 1{\cdot}6267;$$
$$\text{antilog} = 4234;$$
$$\therefore x = £42{\cdot}34 \text{ (about } {\cdot}008 \text{ too great)}$$
$$= £42.\ 6s.\ 10d. \text{ approximately.}$$

EXAMPLES. LXIX.

1. If $\frac{6}{11}$ of a ton cost £45, what is the value of $\frac{13}{15}$ cwt. at the same rate?

2. The cost of $75\frac{5}{6}$ yards of silk is £23. 13*s.* 11½*d.*: find how much can be bought for 10*s.* 11¼*d.* at the same rate.

3. The value of ·075 of a ton of coffee is £61. 12*s.*: find what decimal of 1 lb. of it can be bought for ·458$\dot{3}$ shillings.

4. 3·25 cwts. of goods can be carried $286\frac{2}{13}$ miles for 7·75 shillings: find how far 15·5 cwts. of the same goods should be carried for the same money.

5. The value of $\frac{1}{11}$ of $\frac{1}{12}$ of a ship is £100: find the value of $\frac{3}{11}$ of $\frac{1}{9}$ of $\frac{7}{8}$ of it.

6. If 22½ cwts. be carried 20 miles for £$5\frac{7}{16}$, what weight can be carried the same distance for £14½?

7. The 4*d.* loaf weighs 1 lb. 15½ oz. when wheat is at 7*s.* 11*d.* per bushel: find what its weight should be when wheat is at 7*s.* 1½*d.* per bushel.

8. If the 6*d.* loaf weighs 3 lbs. 5⅓ oz. when wheat is at 6*s.* a bushel, what should it weigh when wheat is at 5*s.* a bushel?

9. If 2 men or 5 boys can do a piece of work in 4 days, in how many days can 1 man and 1 boy together do it?

10. If 8 men or 12 boys can dig a ditch in 20 days, how long will 3 men and 5 boys take to do it?

11. 3 men can do as much as 5 boys, and a work in which 40 boys and 15 men are engaged takes 8 weeks: how long would it take if 20 boys and 20 men were employed upon it?

12. If 5 men or 7 women can do a piece of work in 37 days, how long will a piece of work twice as great occupy 7 men and 5 women?

13. A fortress is provisioned for 3 weeks at the rate of 15 ozs. a day for each man: if only 10½ ozs. be served out daily to each man, how long can the place hold out?

14. An army of 6300 is besieged in a town and has provisions for 10 months: how many men must be sent away that the same provisions may last the remainder 18 months?

15. In a besieged garrison there are 3000 soldiers with provisions for 120 days: how many men must leave the garrison that the provisions may supply the remainder for 180 days?

16. A man-of-war with 1200 troops on board had sufficient provisions to last 17 weeks; the survivors of a wreck having been taken aboard, the provisions were consumed in 15 weeks: how many were taken aboard?

17. A garrison having provisions for 60 days, at the rate of 12 oz. per day per man, was reinforced by three times as many men as it already contained: how long did the provisions last if no reduction was made in the daily allowance?

18. A fortress garrisoned by 100 soldiers had 8 weeks' provisions; but the fortress being besieged, the daily allowance per man was reduced from 18 ozs. to 12 ozs.: how long did the provisions last?

19. A ship leaves port with sufficient provisions to last 14 weeks: 6 of the crew absconded upon setting sail, and the voyage lasted 16 weeks, at the end of which time the provisions were just exhausted: find the number of the full crew.

20. Given that the areas of circles are as the squares of their radii, and that the area of a circle, the radius of which is 10 yds., is 314·159 sq. yds., find the area of a circle the radius of which is 12 yds.

21. At a constant temperature a mass of air occupies 50 cubic feet under a pressure of 15 lbs. to the square inch: what space will it occupy under a pressure of 30 lbs. to the square inch, it being given that, the temperature being the same, the volume occupied by a gas varies inversely as the pressure?

22. The temperature remaining the same, if a mass of air occupies 18 cubic inches under a pressure of $7\frac{1}{2}$ lbs. to the square inch, what space will it occupy under a pressure of 25 lbs. to the square inch, supposing the volume to vary inversely as the pressure?

23. Given that the areas of similar figures are to each other as the squares of their like dimensions, if the area of a rectangular piece of ground is 270 sq. chains and one of its sides 18 chains, find the area of a similar rectangle the corresponding side of which is 24 chains.

24. The areas of similar figures being to each other as the squares of their like dimensions, if a pipe $1\frac{1}{2}$ inches in diameter pours 45 gallons of water into a cistern in a certain time, how much, under the same pressure, will a pipe 2 inches in diameter pour into it in the same time?

25. Given that similar volumes vary as the cubes of their like dimensions, if a ball 3 inches in diameter weighs 9 lbs., find the weight of a ball of the same material the diameter of which is 4 inches.

26. If similar volumes are to each other as the cubes of their like dimensions, and if a cubical box 4 feet long holds 64 bushels of corn, find how much a similar box 6 feet long will hold.

27. The force with which two material particles attract each other varying directly as their masses and inversely as the square of their distances from each other, if two masses at a distance of 20 feet exert an attraction represented by unity, find the attraction at a distance of 1 mile.

Compound Proportion.

So far our discussion of proportion has, in the main, been confined to the consideration of a single rank of two equal ratios. We now proceed to discuss cases in which there are two or more ranks of equal ratios.

Ex. 1. *If* 24 *men can reap* 480 *acres in* 28 *days, how many men can reap* 40 *acres in* 7 *days?*

(1) When the time is constant, the number of workers required is proportional to the amounts of work, less work requiring fewer workers, and more work requiring more workers.

In the example it is less work requiring fewer workers, and fewer in the ratio $\frac{40}{480}$, the ratio of decrease in the work.

(2) When the work is constant, the number of workers is inversely proportional to the times, less time requiring more workers, and more time requiring fewer workers.

In the example it is less time requiring more workers, and more in the ratio $\frac{28}{7}$, the inverse ratio of the times.

Hence the number of workers depends partly on the work and partly on the time, decreasing in the ratio of the amounts of the work and increasing in the inverse ratio of the times.

Taking the two modifying circumstances separately, we have

(1) when the time is constant,

$$\tfrac{40}{480} \text{ of 24 men or 2 men ; and}$$

(2) when the work is constant,

$$\tfrac{28}{7} \text{ of 24 men or 96 men ;}$$

and neither of these results is correct for the whole question, for in neither are the *two* modifying circumstances taken into account.

The first modifying circumstance gives $\frac{40}{480}$ of 24 men, and this result is affected by the second modifying circumstance, which increases it in the ratio $\frac{28}{7}$; hence the second modifying circumstance gives

$$\tfrac{28}{7} \text{ of } \tfrac{40}{480} \text{ of 24 men, that is, 8 men.}$$

$$\text{Now, } \tfrac{28}{7} \times \tfrac{40}{480} = \tfrac{1}{3},$$

$$\text{and 8 men : 24 men} = \tfrac{8}{24} = \tfrac{1}{3}.$$

Hence *the ratio of the men is equal to the ratio compounded of the two modifying ratios.*

Let x represent the required number of men,

$$\text{then } \frac{x}{24} = \frac{40}{480} \times \frac{28}{7};$$

$$\therefore x = 24 \text{ men} \times \frac{40}{480} \times \frac{28}{7}$$

$$= 8 \text{ men.}$$

And, generally, *when a ratio is separately affected by any number of other ratios, it is equal to the ratio compounded of the other ratios.*

Ex. 2. *The rent of a farm of* 41·25 *acres for* 39 *months was* £89·375: *what would be the area of another farm the rent of which for* 33 *months was* £103·125, 4 *acres of the latter being worth as much as* 3 *acres of the former?*

Other things being constant, the area is

(1) inversely proportional to the time, more time giving smaller area and less time larger area; in the present case, larger in the ratio $\frac{39}{33}$, which is the first modifying circumstance;

(2) proportional to the amount of the rent, less rent giving smaller and more rent larger area; in the present case, larger in the ratio $\frac{103 \cdot 125}{89 \cdot 375}$ or $\frac{103125}{89375}$, which is the second modifying circumstance;

(3) inversely proportional to the quality of the land, richer land giving smaller and poorer land larger area; in the present case, larger in the ratio $\frac{4}{3}$, which is the third modifying circumstance.

The total effect of the three modifying circumstances upon the area is therefore represented by the ratio compounded of the ratios $\frac{39}{33}$, $\frac{103125}{89375}$, and $\frac{4}{3}$;

$$\therefore \text{ area} = 41 \cdot 25 \text{ acres} \times \frac{39 \times 103125 \times 4}{33 \times 89375 \times 3}$$
$$= 75 \text{ acres.}$$

Or thus by an equation of ratios:

Let x represent the area in acres,

$$\text{then} \quad \frac{x}{41 \cdot 25 \text{ acres}} = \frac{39 \times 103125 \times 4}{33 \times 89375 \times 3};$$

$$\therefore \text{ as before} \quad x = 41 \cdot 25 \text{ acres} \times \frac{39 \times 103125 \times 4}{33 \times 89375 \times 3}.$$

Ex. 3. *A certain quantity of air occupies* $70\frac{3}{4}$ *cubic inches when the absolute temperature is* 283° *centigrade and the barometer at* 28 *inches: how many cubic inches will it occupy when the absolute temperature is* 273° *centigrade and the barometer at* 30 *inches, it being given that the volume of a gas varies inversely as the pressure and directly as the temperature?*

Other things being constant, the volume is

(1) inversely proportional to the pressure, lower pressure giving greater volume and higher smaller volume; in the present case, smaller in the ratio $\frac{28}{30}$.

(2) proportional to the temperature, higher temperature giving greater and lower temperature smaller volume, smaller in the ratio $\frac{273}{283}$;

$$\therefore \text{ volume} = 70\tfrac{3}{4} \text{ cubic inches} \times \frac{28 \times 273}{30 \times 283}$$
$$= 63 \cdot 7 \text{ cubic inches.}$$

Or thus by an equation of ratios:

Let x represent the volume in cubic inches,

$$\text{then} \quad \frac{x}{70\tfrac{3}{4}\text{ cubic inches}} = \frac{28 \times 273}{30 \times 283};$$

$$\therefore \ x = 70\tfrac{3}{4}\text{ cubic inches} \times \frac{28 \times 273}{30 \times 283} = 63{\cdot}7\text{ cubic inches.}$$

Ex. 4. *If a company of* 160 *men, in* 6 *days of* 11 *hours each, can dig a trench* 230 *yards long,* 5½ *yards wide, and* 1½ *yards deep, in how many days of* 8 *hours each would another company consisting of* 96 *men dig a trench* 220 *yards long,* 3½ *yards wide, and* 1 *yard deep, supposing that the hardness of the ground in the former case is to that in the latter as* 5 *to* 7, *and that* 4 *men of the latter company can do as much work as* 5 *of the former in the same time?*

$$\begin{aligned}\text{No. of days of 8 hours each} &= 6\text{ days} \times \frac{160}{96} \times \frac{11}{8} \times \frac{220}{230} \times \frac{3\frac{1}{2}}{5\frac{1}{2}} \times \frac{1}{1\frac{1}{2}} \times \frac{7}{5} \times \frac{4}{5} \\ &= \frac{6 \times 160 \times 11 \times 220 \times 7 \times 2 \times 7 \times 4}{96 \times 8 \times 230 \times 11 \times 3 \times 5 \times 5} \\ &= 6\tfrac{86}{345}\text{ days or } 6\tfrac{1}{4}\text{ days nearly.}\end{aligned}$$

In the solution of this example, the reasoning is left to the student.

EXAMPLES. LXX.

1. If 4 tons be carried 193 miles for 17*s.* 4*d.*, what will it cost to carry 6 tons 3 cwts. 3 qrs. twice that distance?

2. If £9. 16*s.* be the wages of 14 men for 7 days, what will be the wages of 20 men for 4 days at the same rate?

3. If 35 men consume 52 quarters of wheat in 7 months when wheat is 52 shillings per quarter, what quantity of wheat should they consume in 9 months when wheat is at £3 per quarter, so as not to increase the rate of expense?

4. If 22 loaves last 9 persons 7 days when the loaf costs 1*s.* 8*d.*, how many persons can be fed 12 days at the same cost when the price of the loaf is 7*d.*, the rate of consumption being the same?

5. If 4 men by working 10½ hours a day can mow a field of grass 189 yards long and 160 yards wide in 3 days, how many hours a day must 3 men work to mow a field 275 yards long and 144 yards wide in 5 days?

6. If 20 men can build a wall 800 ft. long, 10 ft. high, and 18 inches thick, in 14 days, working 8 hours a day, how thick a wall 900 ft. long and 15 ft. high, could 15 men build in 21 days, working 9 hours a day?

7. If 18 men can dig a trench 200 yds. long, 3 yds. wide, and 2 yds. deep, in 6 days of 10 hours each, in how many days of 8 hours each will 10 men dig a trench 100 yds. long, 4 yds. wide, and 3 yds. deep?

8. If 12 men build a wall 60 ft. long, 4 ft. thick, and 20 ft. high in 24 days, working 12 hours a day, how many men will it take to build a wall 100 ft. long, 3 ft. thick, and 12 ft. high, in 18 days, working 8 hours a day?

9. If 7 men working 16 days can mow a field of corn 1320 yds. long and 880 yds. wide, what will be the length of the side of a field 1320 yds. wide which 4 men can mow in 42 days?

10. If 150 men can make an embankment 400 yds. long, 12 ft. broad, and 8 ft. high, in 18 days of 9 hours each, in how many days of 8 hours each would 225 men make an embankment 650 yds. long, 10 ft. broad, and $1\frac{1}{2}$ yds. high?

11. If a wall 660 yds. long, 10 ft. high, and $22\frac{1}{2}$ ins. thick, be built by 27 men in $31\frac{1}{4}$ days of 11 hours each, in how many days of 12 hours each could 60 men build a wall 12 ft. high and $2\frac{1}{4}$ ft. thick round a park $4\frac{1}{2}$ miles in circumference?

12. If 48 prisoners in 5 days of $12\frac{1}{2}$ hours long can dig a trench 139·75 yds. long, $4\frac{1}{2}$ yds. wide, and $2\frac{1}{2}$ yds. deep, how many hours per day must 290 prisoners work during 42 days in order to dig a trench $4910\frac{1}{16}$ yds. long, $4\frac{7}{8}$ yds. wide, and $3\frac{1}{5}$ yds. deep?

13. If 5 compositors in 16 days of 14 hours each can compose 20 sheets of 24 pages in a sheet, 50 lines in a page, and 40 letters in a line, in how many days 7 hours long can 10 compositors compose a volume containing 40 sheets of 16 pages to the sheet, 60 lines in a page, and 50 letters in a line?

14. If the wages of 54 men for 36 days amount to £60. 15*s.*, what will be the wages of 30 men and 50 boys for 32 days, supposing 2 men to do as much work as 5 boys?

15. If 5 men can mow 33 acres in 5 days, working 11 hours a day, how many days would it take 4 men to mow 32 acres, working 10 hours a day, if they can mow 5 acres in the same time in which the others can mow 6 acres?

16. If 7 oxen can eat 3 tons of hay in a month, how long will it take 49 sheep to eat 35 tons, an ox consuming 3 times as much as a sheep?

17. If 15 horses and 148 sheep can be kept for 9 days for £75. 15*s.*, what sum will keep 10 horses and 132 sheep for 8 days, supposing 5 horses to eat as much as 84 sheep?

18. If 5 men and 9 boys could do a piece of work in 17 days, in how many days would 9 men and 12 boys do it, the work of 2 men being equal to that of 3 boys?

19. If 42 casks of biscuits, each containing 60 lbs., serve 18 men during a voyage of 16 weeks, how long will 48 casks, each containing 70 lbs., serve 24 men, the daily allowance of the latter crew being $\frac{4}{5}$ of that of the former?

20. If 90 cwt. can be carried 840 English miles for £121, how many Irish miles ought 270 cwt. to be carried for £550, it being given that an Irish mile is 1·27 of an English mile?

21. If 3 men, 4 women, and 5 children can reap a field of 9 acres 2 roods in 6 days, working 10 hours a day, in what time can 7 men, 6 women, and 11 children reap a field of 14 acres 1 rood, working 12 hours a day, supposing the work done by a man, woman, and child to be in the ratio of 3 : 2 : 1?

22. The paving of a yard with tiles 9 inches square cost £8. 1*s.* 4*d.*: what would have been the cost had the tiles been 9 inches long and 6 inches broad, their price being to that of the former as 5 : 8, and the cost of labour being the same?

23. If 25 chests of tea, each containing 168 lbs., cost £306. 5*s.*, how much should be given for 18 chests, each containing 150 lbs., 11 lbs. of the latter being worth 12 lbs. of the former?

24. A farm of 37·5 acres was rented for 36 months for £56·25: for how long, at the same rate, should another farm of 56·25 acres be rented for £50·625, the rent of 5 acres of the latter being equal to that of 4 acres of the former?

25. The rent of a farm of 17·7625 acres is £39. 4*s.*: find the probable rent of another farm of 28·42 acres, if 6 acres of the former be equal in value to 7 acres of the latter?

26. If 17 men by working 8 hours a day made an excavation 121 ft. 6 inches long, 25 ft. 6 inches broad, and 24 ft. deep in 54 days, how many hours daily must 18 men work during 51 days in order that they may make an excavation whose length and breadth are 1 ft. 6 inches less and depth 1 ft. 6 inches greater than the preceding one, supposing that 9 men of the latter do as much as 10 men of the former?

27. If 45 men by working 9 hours a day make an excavation 60 yds. long, 43 ft. 6 inches broad, and 40½ ft. deep in 87 days, how long will it take 58 men, working 8 hours a day, to make an excavation, the length of which is 2 yds. less, and the breadth and depth each 1 yd. less than the preceding one, supposing that 6 men of the former gang do as much work as 7 men of the latter gang?

28. A garrison of 800 men provisioned for 35 days at the rate of 18 ozs. a day per man, receives an augmentation of 250 men: by how much must the daily allowance be diminished that the provisions may last the whole of them for 44 days?

29. A vessel with a crew of 32 men was provisioned for a voyage of 84 days, at the rate of 19½ ozs. a day per man; at the end of 57 days 13 shipwrecked men were picked up, and the vessel was delayed 12 days: find to what each man's allowance must be reduced to make the provisions last to the end of the voyage.

30. A ship is provisioned for a crew of 200 men for 5 weeks, at the rate of 15 ozs. a day per man; if the crew be increased by 100 men and the destination of the ship changed so that the voyage will last 4 weeks only, find the daily allowance per man.

31. A vessel with a crew of 27 men provisioned for 90 days at the rate of 22 ozs. a day per man, was, after 27 days, forced by stress of weather to lie at anchor for a fortnight, at the end of which time 3 men died: how must the provisions be now apportioned that they may hold out the extra time?

32. The force with which the sun attracts the planets varying as their masses and inversely as the square of their distances from the centre of the sun, if the mass of Jupiter is 340 times that of the earth and its distance from the sun's centre is 5 times that of the earth, compare the force with which the sun attracts the earth and Jupiter.

33. The force with which a planet attracts a body on its surface varying as the mass of the body and inversely as the square of its distance from the centre of the planet, if a man weighs 12 stones on the surface of the earth, what would be his weight on the surface of Jupiter, the radius of Jupiter being 11·4 times that of the earth and its mass 340 times that of the earth?

34. The pressure of wind on a plane surface varies jointly as the area of the surface and the square of the velocity of the wind; if the pressure on a square foot is 1 lb. when the wind is moving with a velocity of 15 miles per hour, find the pressure on a square yard when the velocity of the wind is 20 miles per hour.

35. A certain quantity of gas occupies 25 cubic inches at an absolute temperature of 288° centigrade and under a pressure of 29 inches of mercury: determine how many cubic inches it will occupy at an absolute temperature of 293° centigrade and under a pressure of 50 inches of mercury, it being given that the volume of a gas varies directly as the temperature and inversely as the pressure.

CHAPTER XVIII.

DISTRIBUTIVE PROPORTION.

To divide a quantity, say £24, in the ratio of 3 and 5 is so to divide it that for every 3 units in the first there shall be 5 units in the second. Hence, as many times, as the sum of 3 and 5 is contained in 24, so many times can the sum of 3 and 5 be taken from 24.

$$3+5=8, \text{ and } \tfrac{24}{8}=3;$$

∴ we can take from £24 three times £3 and three times £5, that is, £9 and £15.

$$\text{Now,} \quad \tfrac{9}{3}=\tfrac{15}{5}, \text{ for each ratio}=3;$$

$$\text{and} \quad \tfrac{9}{15}=\tfrac{3}{5}, \text{ for each ratio}=\tfrac{3}{5},$$

$$\text{that is,} \quad \frac{\text{1st part}}{\text{1st number}}=\frac{\text{2nd part}}{\text{2nd number}},$$

$$\text{and} \quad \frac{\text{1st part}}{\text{2nd part}}=\frac{\text{1st number}}{\text{2nd number}}.$$

Again, denote the parts by x and y,

$$\text{thus} \quad x:y=3:5,$$

$$\text{and} \quad x:3=y:5;$$

for, as we have seen, *if four quantities are proportionals, they are proportionals when taken alternately*;

$$\therefore \frac{x}{3}=\frac{y}{5}.$$

Now, *when any number of quantities are proportionals, as one antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.*

By the question, sum of antecedents $=x+y=24$,

and sum of consequents $=3+5=8$;

∴ sum of antecedents : sum of consequents $=24:8$;

$$\therefore \frac{x}{3}=\frac{24}{8}, \text{ whence } x=9; \text{ and } \frac{y}{5}=\frac{24}{8}, \text{ whence } y=15.$$

And, generally,

if $a:b=c:d=e:f\ldots,$

then $a:b=a+c+e\ldots:b+d+f\ldots;$

for $\frac{a}{b}=\frac{c}{d}, \quad \therefore ad=bc,$

and $\frac{a}{b}=\frac{e}{f}, \quad \therefore af=be,$

also $ab=ba;$

$$\therefore ab+ad+af\ldots=ba+bc+be\ldots;$$

$$\therefore a(b+d+f\ldots)=b(a+c+e\ldots);$$

$$\therefore \frac{a}{b}=\frac{a+c+e\ldots}{b+d+f\ldots},$$

that is, $a:b=a+c+e\ldots:b+d+f\ldots$

The truth of the theorem may be represented graphically thus:

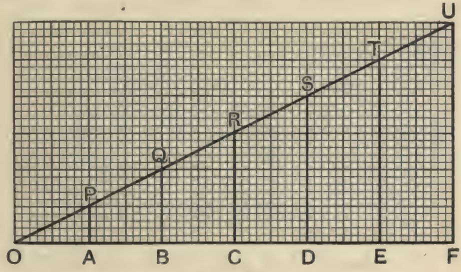


FIG. 13.

It is seen from the graph that

$$PA:OA=1:2,$$
$$QB:OB=2:4=1:2,$$
$$RC:OC=3:6=1:2,$$
$$SD:OD=4:8=1:2,$$
$$TE:OE=5:10=1:2,$$
$$UF:OF=6:12=1:2.$$

$\therefore$ sum of antecedents : sum of consequents $=21:42=1:2$, which is the ratio of any one of the antecedents to its consequent.

Ex. 1. *Divide £45 among three persons in the ratio of* 3, 5, 7.

Denote the parts by x, y, z,

$$\text{then}\quad x:3=y:5=z:7.$$

$$\text{Sum of antecedents}=x+y+z=45,$$

$$\text{sum of consequents}=3+5+7=15;$$

$$\therefore\ \frac{x}{3}=\frac{45}{15},\ \text{whence } x=9,$$

$$\frac{y}{5}=\frac{45}{15},\ \text{whence } y=15,$$

$$\frac{z}{7}=\frac{45}{15},\ \text{whence } z=21,$$

that is, the parts are £9, £15, £21.

The working of such examples is usually set out thus:

$$\text{sum of parts}=3+5+7=15;$$

$$\therefore\ \text{first part} =\tfrac{3}{15}\text{ of £45}=\text{£9},$$

$$\text{second part}=\tfrac{5}{15}\text{ of £45}=\text{£15},$$

$$\text{third part} =\tfrac{7}{15}\text{ of £45}=\text{£21}.$$

Ex. 2. *Divide* £144. 10*s.* *among three persons in the ratio of* $\cdot 3$, $\cdot 33$, $\cdot\dot{3}$.

$$\cdot 3:\cdot 33:\cdot\dot{3}=\tfrac{3}{10}:\tfrac{33}{100}:\tfrac{1}{3}.$$

As we have seen, the ratios that quantities have to each other are not altered when all the quantities or all the ratios are multiplied or divided by any, the same, number. To clear of fractions multiply the ratios by 300.

Then $\tfrac{3}{10}:\tfrac{33}{100}:\tfrac{1}{3}=90:99:100.$

$$\text{Sum of parts}=90+99+100=289;$$

$$\therefore\ \text{share of first}=\tfrac{90}{289}\text{ of £144. 10}s.=\text{£45},$$

$$\text{share of second}=\tfrac{99}{289}\text{ of £144. 10}s.=\text{£49. 10}s.,$$

$$\text{share of third}=\tfrac{100}{289}\text{ of £144. 10}s.=\text{£50}.$$

Ex. 3. *A, B, and C jointly own an estate, their shares being as* $4:2\tfrac{1}{2}:1\tfrac{1}{2}$; *$A$ sells* $\tfrac{1}{2}$ *of his share to C, who sells* 100 *acres to B, and B's share and C's are then equal: how many acres had each of the owners at first?*

$$4:2\tfrac{1}{2}:1\tfrac{1}{2}=8:5:3;$$

$$\therefore\ \text{sum of parts}=16;$$

$$\therefore\ A\text{'s share}=\tfrac{8}{16},\ B\text{'s}=\tfrac{5}{16},\ C\text{'s}=\tfrac{3}{16}.$$

When A has transferred $\tfrac{1}{2}$ of his share to C, their shares stand at $\tfrac{4}{16}$ and $\tfrac{7}{16}$ respectively.

C now sells 100 acres to B;

$$\therefore\ C\text{'s share now} = \tfrac{7}{16} \text{ of the estate} - 100 \text{ acres},$$
$$\text{and } B\text{'s}\quad ,, \quad ,, \quad = \tfrac{5}{16} \text{ of the estate} + 100 \text{ acres};$$

and B's share and C's are now equal.

Put x for the number of acres in the estate;

then
$$\tfrac{7}{16}x - 100 \text{ acres} = \tfrac{5}{16}x + 100 \text{ acres};$$
$$\therefore\ \tfrac{7}{16}x - \tfrac{5}{16}x = 200 \text{ acres};$$
$$\therefore\ \tfrac{2}{16}x = 200 \text{ acres};$$
$$\therefore\ x = \frac{200 \text{ acres} \times 16}{2} = 1600 \text{ acres}.$$

$$\therefore\ A\text{'s original share} = \tfrac{8}{16} \text{ of } 1600 \text{ ac.} = 800 \text{ ac.},$$
$$B\text{'s}\quad ,, \quad ,, \quad = \tfrac{5}{16} \text{ of } 1600 \text{ ac.} = 500 \text{ ac.},$$
$$C\text{'s}\quad ,, \quad ,, \quad = \tfrac{3}{16} \text{ of } 1600 \text{ ac.} = 300 \text{ ac.}$$

Ex. 4. *Divide* 163 *crowns amongst A, B, and C, so that A's share may be to B's share as* $\frac{1}{2}$ *to* $\frac{3}{4}$, *and B's share to C's share as* $\frac{5}{6}$ *to* $\frac{7}{8}$.

$$A\text{'s share} : B\text{'s} = \tfrac{1}{2} : \tfrac{3}{4} = 2 : 3 \text{ (both terms} \times 4),$$
$$B\text{'s share} : C\text{'s} = \tfrac{5}{6} : \tfrac{7}{8} = 20 : 21 \text{ (both terms} \times 24).$$

In the first proportion B's share is, in relation to A's, denoted by 3; in the second proportion it is, in relation to C's, denoted by 20. If we find the L.C.M. of 3 and 20, we can express the value of B's share by the same number in both.

The L.C.M. of 3 and 20 is 60, where 3 is multiplied by 20 and 20 by 3. Hence we have

$$A\text{'s share} : B\text{'s} = 2 : 3 = 40 : 60 \text{ (both terms} \times 20),$$
$$B\text{'s share} : C\text{'s} = 20 : 21 = 60 : 63 \text{ (both terms} \times 3);$$
$$\therefore\ A\text{'s share} : B\text{'s} : C\text{'s} = 40 : 60 : 63.$$
$$\text{Sum of parts} = 40 + 60 + 63 = 163;$$
$$\therefore\ A\text{'s share} = \tfrac{40}{163} \text{ of } 163 \text{ crowns} = 40 \text{ crowns},$$
$$B\text{'s share} = \tfrac{60}{163} \text{ of } 163 \text{ crowns} = 60 \text{ crowns},$$
$$C\text{'s share} = \tfrac{63}{163} \text{ of } 163 \text{ crowns} = 63 \text{ crowns}.$$

Or thus by a more generalised method:

$$\text{Since}\quad \frac{B\text{'s share}}{C\text{'s share}} = \frac{20}{21};\quad \therefore\ \frac{C\text{'s share}}{B\text{'s share}} = \frac{21}{20};$$
$$\therefore\ C\text{'s share} = \tfrac{21}{20} \text{ of } B\text{'s share},$$
$$\text{and}\quad A\text{'s share} = \tfrac{2}{3} \text{ of } B\text{'s share},$$
$$\text{hence}\quad A\text{'s share} + B\text{'s} + C\text{'s} = \left(\tfrac{2}{3} + 1 + \tfrac{21}{20}\right) \text{ of } B\text{'s share},$$

where 1 represents B's share.

The student can easily finish the solution for himself.

Ex. 5. *Divide £3174 among A, B, C, and D, so that A's share may be to B's as* 6 : 5, *B's share to C's as* 4 : 3, *and C's share to D's as* 3 : 2.

$$A\text{'s share} : B\text{'s} = 6 : 5 = 24 : 20 \text{ (both terms} \times 4),$$
$$B\text{'s share} : C\text{'s} = 4 : 3 = 20 : 15 \text{ (both terms} \times 5),$$
$$C\text{'s share} : D\text{'s} = 3 : 2 = 15 : 10 \text{ (both terms} \times 5).$$

The ratios of B's share to A's and C's are now expressed by the same number (20), and the ratios of C's share to B's and D's are also expressed by the same number (15). Hence we have

$$A\text{'s share} : B\text{'s} : C\text{'s} : D\text{'s} = 24 : 20 : 15 : 10.$$
$$\text{Sum of parts} = 24 + 20 + 15 + 10 = 69;$$
$$\therefore\ A\text{'s share} = \tfrac{24}{69} \text{ of } £3174 = £1104,$$
$$B\text{'s share} = \tfrac{20}{69} \text{ of } £3174 = £920,$$
$$C\text{'s share} = \tfrac{15}{69} \text{ of } £3174 = £690,$$
$$D\text{'s share} = \tfrac{10}{69} \text{ of } £3174 = £460.$$

Or thus by a more generalised method:

$$\frac{A\text{'s share}}{B\text{'s share}} = \frac{6}{5}, \quad \frac{B\text{'s share}}{C\text{'s share}} = \frac{4}{3}, \quad \frac{C\text{'s share}}{D\text{'s share}} = \frac{3}{2}.$$

Since $\dfrac{C\text{'s share}}{D\text{'s share}} = \dfrac{3}{2}$;

$$\therefore\ C\text{'s share} = \tfrac{3}{2} \text{ of } D\text{'s share};$$
$$\therefore\ \frac{B\text{'s share}}{\tfrac{3}{2} \text{ of } D\text{'s share}} = \frac{4}{3};$$
$$\therefore\ B\text{'s share} = \tfrac{4}{3} \text{ of } \tfrac{3}{2} \text{ of } D\text{'s share} = 2 \text{ of } D\text{'s share};$$
$$\therefore\ \frac{A\text{'s share}}{2 \text{ of } D\text{'s share}} = \frac{6}{5};$$
$$\therefore\ A\text{'s share} = \tfrac{6}{5} \text{ of } 2 \text{ of } D\text{'s share} = \tfrac{12}{5} \text{ of } D\text{'s share};$$

hence sum of shares of all expressed in terms of D's share

$$= \left(\tfrac{12}{5} + 2 + \tfrac{3}{2} + 1\right) \text{ of } D\text{'s share} = \tfrac{69}{10} \text{ of } D\text{'s};$$
$$\therefore\ \tfrac{69}{10} \text{ of } D\text{'s share} = £3174;$$
$$\therefore\ D\text{'s share} = £\tfrac{3174 \times 10}{69} = £460;$$
$$\therefore\ A\text{'s share} = \tfrac{12}{5} \text{ of } £460 = £1104;$$
$$B\text{'s share} = £460 \times 2 = £920;$$
$$\text{and } C\text{'s share} = \tfrac{3}{2} \text{ of } £460 = £690.$$

Ex. 6. *A, B, and C entered into partnership for* 1 *year; their capital was* £8000, *of which A subscribed* £3600, *B* £2400, *and C* £2000; *their net profit for the year was* £3000: *find each partner's share of the profits.*

Each partner's capital was employed for the same length of time; hence it is manifest that a partner's share of the profits has the same ratio to the total profits that the capital subscribed by that partner has to the total capital;

$$\therefore\ \frac{A\text{'s profit}}{£3000}=\frac{3600}{8000};\quad \therefore\ A\text{'s profit}=£\frac{3000\times3600}{8000}=£1350;$$

$$\frac{B\text{'s profit}}{£3000}=\frac{2400}{8000};\quad \therefore\ B\text{'s profit}=£\frac{3000\times2400}{8000}=£900;$$

$$\frac{C\text{'s profit}}{£3000}=\frac{2000}{8000};\quad \therefore\ C\text{'s profit}=£\frac{3000\times2000}{8000}=£750.$$

The following method is much briefer.

$$A\text{'s part of the capital}=\tfrac{3600}{8000}=\tfrac{9}{20},$$
$$B\text{'s part of the capital}=\tfrac{2400}{8000}=\tfrac{3}{10},$$
$$C\text{'s part of the capital}=\tfrac{2000}{8000}=\tfrac{1}{4};$$
$$\therefore\ A\text{'s share of profits}=\tfrac{9}{20}\text{ of }£3000=£1350,$$
$$B\text{'s share of profits}=\tfrac{3}{10}\text{ of }£3000=£900,$$
$$\text{and }C\text{'s share of profits}=\tfrac{1}{4}\text{ of }£3000=£750.$$

Ex. 7. *A and B enter into partnership, the former putting in* £400 *for* 3 *months and the latter* £750 *for* 4 *months; if they gain* £700, *what is each man's share of the profits?*

£400 for 3 months = £400 × 3, or £1200, for 1 month;

£750 for 4 months = £750 × 4, or £3000, for 1 month;

hence a capital of £400 for 3 months + a capital of £750 for 4 months = a capital of (£1200 + £3000) for 1 month;

∴ the equivalent capitals for 1 month are *A*'s £1200, *B*'s £3000, total £4200.

$$\therefore\ A\text{'s part of total capital}=\tfrac{1200}{4200}=\tfrac{2}{7};$$
$$\text{and }B\text{'s part of total capital}=\tfrac{3000}{4200}=\tfrac{5}{7};$$
$$\therefore\ A\text{'s share of profits}=\tfrac{2}{7}\text{ of }£700=£200,$$
$$\text{and }B\text{'s share of profits}=\tfrac{5}{7}\text{ of }£700=£500.$$

Ex. 8. *Three men rent a field for* £80; *the first put into it* 56 *horses for* 12 *days, the second* 64 *for* 15 *days, and the third* 80 *for* 18 *days: find each one's share of the rent.*

56 horses for 12 days = 56 × 12, or 672 for 1 day;

64 horses for 15 days = 64 × 15, or 960 for 1 day;

80 horses for 18 days = 80 × 18, or 1440 for 1 day;

∴ 56 horses for 12 days + 64 for 15 days + 80 for 18 days

= (672 + 960 + 1440) horses for 1 day

= 3072 horses for 1 day;

∴ the ratio of the first man's horses to the whole

$$=\tfrac{672}{3072}=\tfrac{7}{32},$$

the ratio of the second man's horses to the whole

$$=\frac{960}{3072}=\frac{5}{16},$$

and the ratio of the third man's horses to the whole

$$=\frac{1440}{3072}=\frac{15}{32};$$

$\therefore$ the first man's share of rent $=\frac{7}{32}$ of £80 = £17. 10*s.*,

the second man's share $=\frac{5}{16}$ of £80 = £25,

and the third man's share $=\frac{15}{32}$ of £80 = £37. 10*s.*

Ex. 9. *A, B, and C are partners; A's money has been in the business 3 months and he claims $\frac{1}{12}$ of the gain; B's money has been in the business 9 months, and C has had £756 in the business 4 months, and he claims half the gain: how much money did A and B contribute to the capital?*

A's capital × 3 = *A*'s equivalent capital for 1 month;

B's capital × 9 = *B*'s equivalent capital for 1 month;

£756 × 4 = *C*'s equivalent capital for 1 month;

$\therefore$ £3024 = *C*'s equivalent capital for 1 month.

Now *C* claims half the profit;

$\therefore$ he must have subscribed half the equivalent capital;

$\therefore$ total equivalent capital = £3024 × 2 = £6048.

A claims $\frac{1}{12}$ of the profit;

$\therefore$ $\frac{1}{12}$ of £6048 = *A*'s equivalent capital = £504.

A claims $\frac{1}{12}$ and *C* $\frac{1}{2}$ of the profit;

$\therefore$ *B*'s share of profit $=\frac{5}{12}$;

$\therefore$ $\frac{5}{12}$ of £6048 = *B*'s equivalent capital = £2520;

but *A*'s equivalent capital = 3 times his capital;

$\therefore$ *A*'s capital $=£\frac{504}{3}=£168$;

and *B*'s equivalent capital = 9 times his capital;

$\therefore$ *B*'s capital $=£\frac{2520}{9}=£280$.

EXAMPLES. LXXI.

1. In a mixed school of 260 children the boys are to the girls as 3 to 2: find the number of each sex in the school.

2. Divide £140 among 3 persons, so that their shares may be to each other as 1, 2, and 3.

3. Divide £12012 among 3 persons in the ratio of 2, $5\frac{1}{3}$, and $5\frac{1}{2}$.

4. Distribute £80. 15*s.* among *A*, *B*, and *C* in the ratio of 1, $1\frac{1}{2}$, and $2\frac{1}{2}$ respectively.

5. Divide 80 guineas among A, B, and C in the ratio of $7\frac{1}{3}$, $10\frac{1}{3}$, and $14\frac{1}{3}$ respectively.

6. Divide £11000 among 4 persons in the ratio of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$.

7. A man bequeaths his property to his four sons, whose shares are in the ratio of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$: if the share of the youngest is £900, what are the shares of the others?

8. The sum of £7575 is to be divided among 4 boys and 5 girls, and the share of each boy to that of each girl is to be as $\frac{2}{3}$ to $\frac{3}{7}$: find the share of each boy and of each girl.

9. The sum of two numbers is 342, and one is to the other as 2^5 to 5^2: find the numbers.

10. A purse contains £42. 8*s.*, made up of pence, shillings, half-crowns, and sovereigns; the half-crowns, pence, sovereigns and shillings are as the numbers 1, 2, 3, and 8 respectively: find how many of each coin are in the purse.

11. Divide £300 among A, B, and C, so that A's share may be to B's as 2 to 3, and B's to C's as 2 to 5.

12. £1630 is divided among A, B, and C; A's share : B's share is as $\frac{1}{2} : \frac{3}{4}$, and B's : C's as $\frac{5}{6} : \frac{7}{8}$: find the share of each.

13. The sum of £1908. 18*s.* is divided among A, B, and C; A's share : B's $= 3 : 5$, and B's : C's $= 10 : 11$: find the share of each.

14. If £3728. 7*s.* 6*d.* is divided among A, B, and C in such a way that A's share is to B's as 1 to $1\frac{1}{2}$, and B's to C's as 1 to $1\frac{1}{4}$, find the share of each.

15. Divide £416. 6*s.* among A, B, C, and D, so that A's share may be to B's as 5 to 4, B's to C's as 4 to 3, and C's to D's as 9 to 10.

16. Divide £1478, 15*s.* among A, B, C, and D, so that A's share : B's $= 3 : 8$, B's : C's $= 4 : 9$, and C's : D's $= 15 : 4$.

17. Three partners buy a ship; the first contributing £13500, the second £11000, and the third £8500: if she earns £7986 on her first voyage, how much of this ought each of the partners to receive?

18. The profits for the year of two partners, A and B, amount to £1820; A put £300 and B £400 into the business: find each one's share of the profits.

19. A and B enter into partnership, A contributing £800, and B £1000: at the end of 6 months they admit C, who contributes £600: after the lapse of 3 years they balance the books, discovering a profit of ££966: find the share of each.

20. In a certain speculation A had invested £4000 for 1 year, B £3000 for 15 months, and C £5000 for 8 months: if the profits were £665, how much was each persons's share?

21. A merchant commenced business with £3500; at the end of 6 months he was joined by a partner who brought £14000, and at the end of 12 months by a second partner who brought £7000 into the business; at the end of the third year the profits amounted to £9817. 10*s.*: find each partner's share of the profits.

22. *A* contributed £50, *B* £60, and *C* £80 to a business; *B* withdrew his money at the end of 10 months, and *A* and *C* continued the business for 6 months longer, when the profits were £402; find each partner's share of this gain.

23. *A*, *B*, and *C* pay £40 as the year's rent of a farm; *A* puts 40 cows on it for 6 months, *B* 30 cows for 5 months, and *C* 50 cows for the rest of the time: how much of the rent should each person pay?

24. Three persons rent a field for £121; the first puts in 10 sheep for $4\frac{1}{2}$ months, the second 16 sheep for 5 months, and the third 18 sheep for $6\frac{1}{2}$ months: find each one's share of the rent.

25. *A*, *B*, and *C* rent a field jointly for £25. 12*s.* 5*d.*; *A* puts in 15 sheep for 6 months, *B* 45 sheep for 8 months, and *C* 81 sheep for 4 months: what should each pay towards the rent?

26. *A*, *B*, and *C* are partners; *A*'s money has been in the business 3 months, and he claims $\frac{1}{12}$ of the gain; *B*'s money has been in the business 9 months, and *C* has had £756 in the business 4 months, and he claims half the gain: how much money did *A* and *B* contribute to the capital?

27. *A*, *B*, and *C* are partners; *A*'s money has been in the business 16 months, and he claims $\frac{20}{67}$ of the profit; *B*'s money has been in it 10 months, and he claims $\frac{15}{67}$ of the profit; and *C* has had £80 in it 16 months: find how much capital *A* and *B* have in the business.

28. *A*, *B*, and *C* rent a farm for £270; *A* puts 200 sheep on it, *B* 150, and *C* 100; after 6 months *A* sells $\frac{3}{4}$ of his flock to *C*, and 3 months later *B* sells $\frac{2}{3}$ of his to *A*: how much of the rent should each pay at the end of the year?

29. *A* and *B* enter into partnership; *A* puts in £2100 and *B* £1500; four months after *C* enters the partnership with a capital of £2700; at the end of the year the profits were found to be 10 per cent. on the whole capital invested: how much of the profits should each partner receive?

30. *A* and *B* enter into partnership; *A* puts into the business £1500 more than *B*, who, as acting partner, is to have a salary of £150 a year for management; at the end of 3 years the gross profits, computed at 15 per cent. per annum, are found to be £2925, from which *B*'s salary is to be deducted: it is required to find the amount of capital subscribed by each partner and also his share of the net profit.

31. *A* and *B* hold ground in common at a rental of £54; *A* puts in 23 horses for 27 days, and *B* 21 horses for 39 days: how much of the rent should be paid by each?

32. Four persons enter into partnership: the second has twice as large a capital as the first, and the third as much as half the sum of the capital of the first two, and the fourth has a sum equal to the capital of all the others: if the total profits for a period are £10000, how much should each partner receive?

33. *A* and *B* enter into partnership for a year: *A* contributes £500 on the 1st of January, and *B* on the 1st of May contributes as much as entitles him to half the profits at the end of the year: find *B*'s contribution.

34. In a factory there are 1300 workmen placed under four superintendents *A*, *B*, *C*, *D*; for every 4 men under *A* there are 5 under *C*, and for every 9 men under *B* there are 10 under *D*, and for every 2 men under *A* there are 3 under *B*: how many men are there under each?

35. Three persons *A*, *B*, *C*, jointly own an estate, their shares being in the ratio of 4, $2\frac{1}{2}$, and $1\frac{1}{2}$; *A* sells half his share to *C*, and *C* sells 100 acres to *B*, and the shares of *B* and *C* are then equal: find the number of acres in the original shares.

36. A sum of £38. 4*s*. is made up of half-sovereigns, half-crowns, and threepenny pieces; there are twice as many half-crowns as half-sovereigns, and the number of half-sovereigns bears to the number of threepenny pieces the ratio of 3 to 11: how many are there of each coin?

37. A sum of £52. 18*s*. is made up of half-sovereigns, half-crowns, and sixpences, the number of these coins being in the ratio of 3, 5, 7 respectively: how many are there of each coin?

38. *A*, *B*, and *C*'s shares are as $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$; *A* withdraws half his capital at the end of 4 months, and 8 months after the profits, £1056, are divided: find each partner's share.

39. *A* contributed £600 for 5 months to a business and received $\frac{2}{5}$ of the whole profit; and *B* contributed £500: how long was *B*'s money in the business?

40. Three men subscribe £4600; *A* gives £400 more than *B*, who gives £300 more than *C*: if their gain be £345, find what each contributes and how much of the profits each is entitled to.

41. If 1 lb. troy of metal containing copper and zinc in the ratio of ·84 to ·16 be mixed with 2 lbs. containing the same metals in the ratio of ·75 to ·25, find how much copper and zinc the mixture contains.

42. Supposing the weights of pure gold and of alloy in a sovereign to be in the ratio of 113 to 10, and the price of pure gold to be £3. 17*s*. $10\frac{1}{2}$*d*. per oz., find the weight of a sovereign, supposing its value to be that of the pure gold it contains.

43. Supposing the pure silver in the silver coin of England to be to the copper as 37 to 3, and a lb. troy of the alloy to be coined into 66 shillings, find the weight of pure silver contained in a shilling.

CHAPTER XIX.

GRAPHS.

THE exact position of a point in a plane is most easily found by measuring its distance from two lines drawn at right angles to each other.

Let Fig. 14 represent a rectangular garden, and the points marked *a*, *b*, *c* represent points in the garden. Then it is plain that we can locate the position of any one of the points by finding its distance from any two of the bounding lines of the figure.

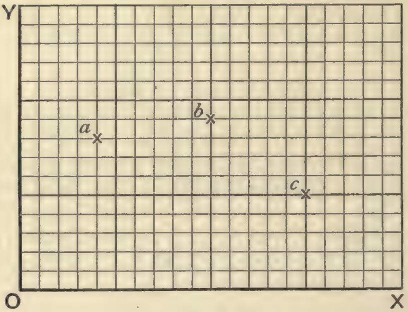


FIG. 14.

Take the sides OX and OY, and let each of the small squares represent a square yard. Then $OX = 20$ yards, and $OY = 15$ yards; and the point a is seen to be 4 yards from OY, and 8 yards from OX.

It is plain that, even if there were no horizontal line connecting the point a with the line OY, we could find the position of the point by measuring 4 yards along OX and then 8 yards vertically up from OX. The point a in the plane of the lines OX and OY is therefore determined by the numbers 4, 8.

In a similar way the point b is fixed by the numbers 10, 9, and the point c by the numbers 15, 5.

It is usual to denote measurements along OX by x, and those along OY by y.

The lines OX and OY are called *lines of reference* or *axes of reference*, OX being known as the *x-axis* and OY as the *y-axis*; and the point O, where they intersect, is called the *origin*.

Measurements along OX or parallel to OX are called *abscissae*, and measurements along OY or parallel to OY are called *ordinates*. The abscissa and ordinate that locate the position of a point in the plane of the axes of reference are called *co-ordinates* of the point.

It is plain from the figure

(1) *that the co-ordinates of the origin are* (0, 0),

(2) *that, if a point lies in the x-axis, its ordinate is* 0,

(3) *that, if a point lies in the y-axis, its abscissa is* 0.

In the notation of graphs, abscissae are named before ordinates. Thus a is the point (4, 8), b is the point (10, 9), and c is the point (15, 5), and the point (x, y) is a point that is x units of length from OY and y units of length from OX.

It is obvious from what has been said that (4, 8) are the co-ordinates of the point a, (10, 9) the co-ordinates of the point b, and (15, 5) the co-ordinates of the point c.

The process of finding the position of a point from given co-ordinates is called *plotting the point*. As we shall see, the use of squared paper makes the process very easy.

In Fig. 15 the axes $X'OX$ and $Y'OY$ are seen to divide the plane into four compartments or quadrants. All steps, that is, abscissae, measured to the right of the axis $Y'OY$ are considered *positive*, and all steps to the left of it are considered *negative*; and all steps, that is, ordinates, measured upwards from the axis $X'OX$ are considered *positive*, and all steps measured downwards from it are considered *negative*. The negative sign indicates *direction* from the axes. Thus the point P' is denoted by the co-ordinates (5, 6), the point P^2 by the co-ordinates $(-5, 6)$, the point P^3 by the co-ordinates $(-5, -6)$, and the point P^4 by the co-ordinates $(5, -6)$. Hence we can tell at once from the signs of the co-ordinates the quadrant in which a point lies, for in the first quadrant their signs are +, +, in the second they are −, +, in the third they are −, −, and in the fourth they are +, −. It follows therefore that, abscissae being denoted by x and ordinates by y, the point (x, y) is in the first quadrant, the point $(-x, y)$ is in the second quadrant, the point $(-x, -y)$ is in the third quadrant, and the point $(x, -y)$ is in the fourth quadrant.

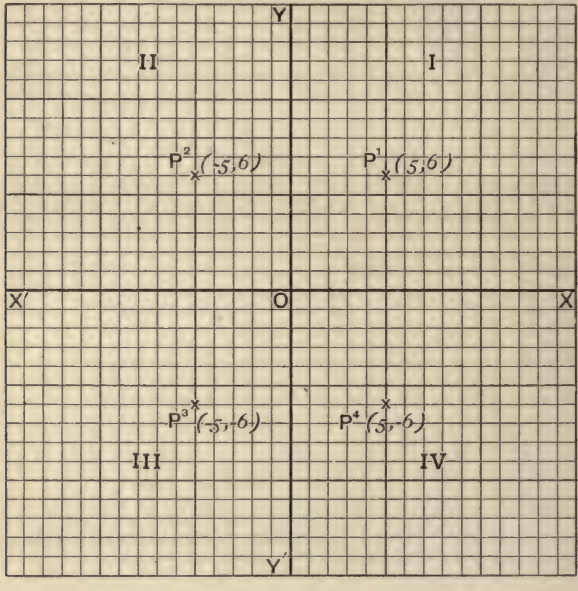


FIG. 15.

EXAMPLES. LXXII.

Plot the points in examples 1-12.

1. (4, 2).
2. (6, 4).
3. (10, 7).
4. (0, 6).
5. (0, 0).
6. (−3, 4).
7. (−3, −4).
8. (−6, 7).
9. (−6, −7).
10. (6, −7).
11. (−8, −12).
12. (8, −12).

The following sets of four points are at the ends of the bounding lines of rectangles: find the areas of the rectangles. (Work on paper ruled in tenths of an inch.)

13. (0, 5), (20, 5), (0, 10), (20, 10).

14. (0, 4), (18, 4), (0, 9), (18, 9).

15. (−7, 2), (−7, 10), (9, 2), (9, 10).

16. (−8, −3), (−8, −15), (7, −3), (7, −15).

17. Plot the points (5, 13) and (10, 0), join them by a straight line, and with your compasses find the length of the line [$\frac{1}{10}$ in. being the unit].

18. Plot the point (8, 15), join it by a straight line to the origin and with your compasses find the length of the line [$\frac{1}{10}$ in. being the unit].

A line (straight or curved) drawn through a series of points in the plane of the axes of reference is called a *graph*, and locating the points and joining them by a line is called *drawing or plotting the graph of the points.*

In the following table the numbers opposite x and y denote the co-ordinates of a series of three points, viz., the points (2, 3), (4, 6), and (8, 12).

Read thus: when $x=2$, $y=3$; when $x=4$, $y=6$, and so on.

$x=$	2	4	8
$y=$	3	6	12

The graph is shown in Fig. 16.

Notice here that each ordinate is $1\frac{1}{2}$ times the corresponding abscissa, for $3=2\times1\frac{1}{2}$, $6=4\times1\frac{1}{2}$, and $12=8\times1\frac{1}{2}$; that is, the ordinates have to the abscissae the ratio of $1\frac{1}{2}$ to 1, this is expressed by the equation, graph of $y=1\frac{1}{2}x$.

Show that no point in the plane of the axes that does not satisfy this equation lies on the graph.

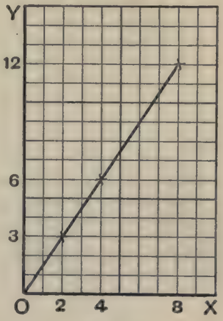


FIG. 16.

Every graph of an equation of the form $y=ax$, where a is a constant factor, is a straight line passing through the origin. That is, in all cases of direct proportionality between two quantities the graph that exhibits their variations is a straight line passing through the origin. This the student will find further illustrated in Fig. 17 and Fig. 18 below.

We can locate the position of any straight line by determining the positions of its ends or of any two points in the line, for only one straight line can be drawn through two given points. Hence, to draw a straight line, only two points on it are needed; but these should be as far apart as possible, for slight inaccuracies in plotting points are apt to cause serious displacement of the line. The origin and one other point are sufficient when the equation is known to be of the form $y = ax$.

EXAMPLES.

Plot graphs from the following data:

1.

$x=$	3	6	9	etc.
$y=$	4	8	12	etc.

Equations:

Graph of $y=1\frac{1}{3}x$.

2.

$x=$	2	4	6	etc.
$y=$	1	2	3	etc.

Graph of $y=\frac{1}{2}x$.

3.

$x=$	5	8	10	etc.
$y=$	$12\frac{1}{2}$	20	25	etc.

Graph of $y=2\frac{1}{2}x$.

In the application of the method of co-ordinates, it is not necessary that x and y should refer to quantities of the same kind, nor that the unit steps of the ordinates and abscissae should be of the same length; for the relation between the co-ordinates is numeric, not quantitive. Hence, if in the same diagram we use throughout the same unit for the abscissae, it does not matter whether it is or is not of the same length as the unit used for the ordinates.

Ex. 1. *Let it be required to exhibit graphically the relation between the cost and the quantity of tea purchased at* 1*s.* 6*d. per lb.*

Construct a table of co-ordinates:

Weight (x) in lbs.	1	2	3	4	5	6	etc.
Price (y) in shillings	$1\frac{1}{2}$	3	$4\frac{1}{2}$	6	$7\frac{1}{2}$	9	etc.

Equation:

Graph of $y=1\frac{1}{2}x$.

It will be sufficient to plot the points (2, 3) and (6, 9), or (0, 0) and (6, 9).

The ordinate of any point on the graph will give the number of shillings corresponding with the number of lbs. indicated by the abscissa of the same point. For example, the point *a* indicates that the cost of 2 lbs. is 3*s.*, the point *b* that the cost of 3 lbs. is 4*s.* 6*d.*, and the point *c* that the cost of 5½ lbs. is 8*s.* 3*d.* The student should read off the cost of other quantities.

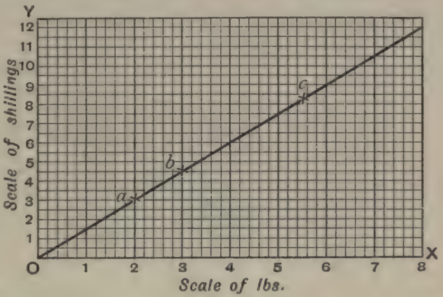


FIG. 17.

Ex. 2. *Exhibit graphically the relation between the number of half-crowns and the number of shillings in any sum of money.*

Table of co-ordinates:

Florins (x)	5	10	etc.
Half-crowns (y)	4	8	etc.

Equation:

5 florins = 4 half-crowns;

$\therefore$ 1 florin = $\frac{4}{5}$ of a half-crown;

$\therefore$ graph of $y = \frac{4}{5}x$.

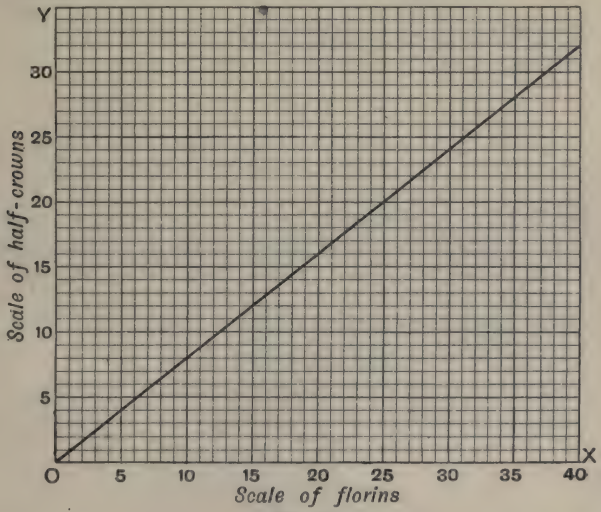


FIG. 18.

Find on the graph the points that indicate the same values in both coins.

Ex 3. *Construct a graph to show the conversion of degrees Fahrenheit to degrees Centigrade.*

The freezing and the boiling point on the Fahrenheit scale are 32° and 212° respectively, and the corresponding points on the Centigrade scale are 0° and 100° respectively.

Hence
$$100^\circ\text{ C.} = (212^\circ - 32^\circ)\text{ F.}$$
$$= 180^\circ\text{ F.};$$
$$\therefore\ 1^\circ\text{ C.} = \frac{180^\circ}{100}\text{ F.} = \frac{9^\circ}{5}\text{ F.}$$

Let y denote the number of degrees on the F. scale, and x the number of degrees on the C. scale; then

$$y = \tfrac{9}{5}x + 32.$$

The graph of an equation of the form $y = ax + b$, where a and b are constants, is always a straight line passing, not through the origin, but through a point in the y-axis b units above the origin. In our equation, b is 32; hence the graph begins in the y-axis at a point 32 units above the origin.

From the equation $y = \frac{9}{5}x + 32$, we see that

when $x = 0$, $y = 32$;
when $x = 5$, $y = 9 + 32 = 41$;
when $x = 10$, $y = 18 + 32 = 50$;
when $x = 15$, $y = 27 + 32 = 59$; and so on.

Table of co-ordinates:

$C\ (x)$	0	5	10	15	20	etc.
$F\ (y)$	32	41	50	59	68	etc.

The graph of Fig. 19 is seen to be a straight line; hence its equation is either of the form $y = ax$ or of the form $y = ax + b$, where a and b are constants. If its co-ordinates are in direct proportion, the equation is $y = ax$; but they are not, for $\frac{5}{41}$ is not equal to $\frac{10}{50}$; hence the equation is $y = ax + b$.

Let it be required to find the values of the constants a and b. Take two pairs of corresponding values of x and y; then,

$$\text{since}\quad y = ax + b;$$
$$\therefore\ 41 = 5a + b,$$
$$\text{and}\ \ 50 = 10a + b$$
$$\therefore\ \text{by subtraction}\quad 9 = 5a;$$

$$
\begin{aligned}
\therefore \quad a &= \tfrac{9}{5}\,; \\
\therefore \quad 50 &= \tfrac{9}{5} \times 10 + b \\
&= 18 + b\,; \\
\therefore \quad b &= 32.
\end{aligned}
$$

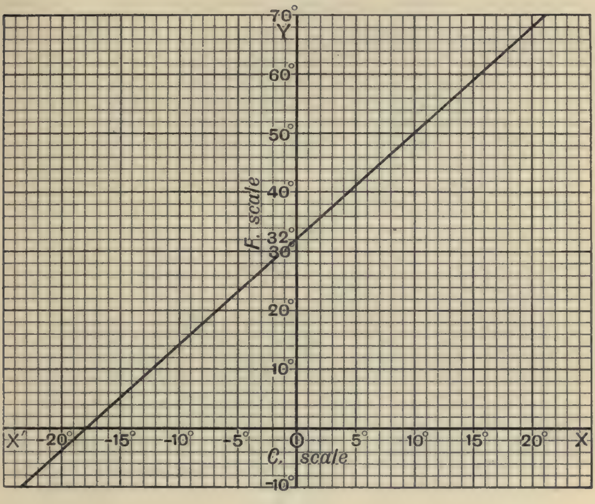


FIG. 19.

Thus, when the co-ordinates of the points of a straight line graph are found or given, it is easy to determine the equation of the graph. In the application of graphs to experimental and observational work, the determination of the equation is of great importance, for it is the equation that shows the relation of the variables x and y.

Ex. 4. *Construct a graph that will enable you to convert inches to centimetres or centimetres to inches,* 1 *inch being equal to* 2·5 *centimetres.*

1 inch = 2·5 centimetres;

∴ 2 inches = 5 centimetres.

Let x denote the number of inches, and y the number of centimetres;

$$
\begin{aligned}
&\text{then} \quad y = \tfrac{5}{2}x\,; \\
\therefore\ &\text{when} \quad x = 2, \quad y = 5\,; \\
&\text{when} \quad x = 4, \quad y = 10\,; \\
&\text{when} \quad x = 6, \quad y = 15\,; \text{ and so on.}
\end{aligned}
$$

Table of co-ordinates:

Inches (x)	2	4	6	etc.
Centimetres (y)	5	10	15	etc.

Equation: Graph of $y=\frac{5}{2}x$.

Fig. 20 exhibits the graph.

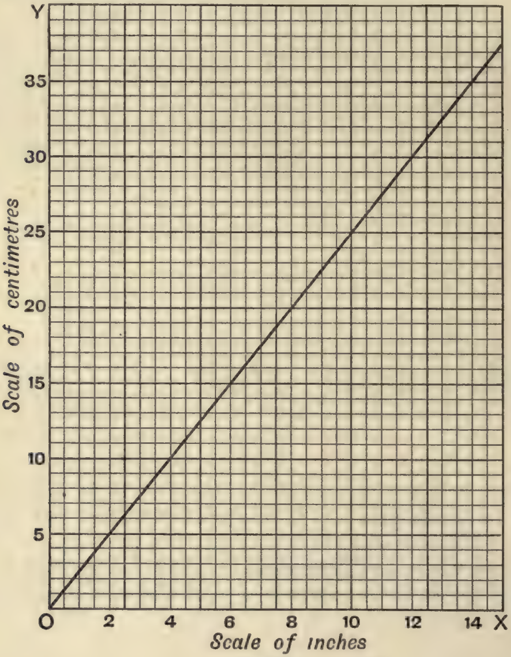


FIG. 20.

Ex. 5. *Construct a graph by means of which speeds expressed in miles per hour can be expressed in feet per second.*

Obviously, the equation of the graph is $y=ax$, for speed per minute is directly proportional to speed per hour; hence the graph passes through the origin, and therefore one point on it is (0, 0).

30 miles per hour is equal to 44 feet per second; hence another point on the graph is (30, 44). Having these data, we plot the points and draw the graph as on Fig. 21.

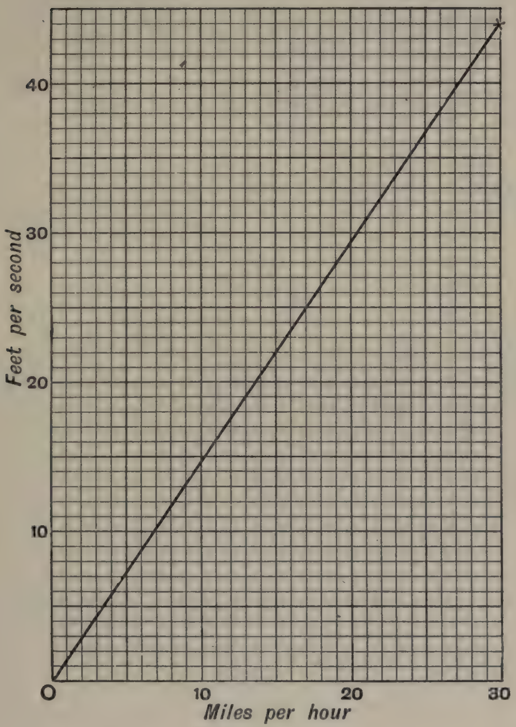


FIG. 21.

Graphical Representation of Statistics.

When statistics are represented by graphs, it is usual to join the plotted points either by straight lines or by a continuous freehand curve.

Ex. 6. *In the following table are given for each month of the year the mean temperature in degrees Centigrade: exhibit graphically the variation of the temperature throughout the year:*

Time	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Jan.	Feb.	M'ch.	April
Temperature	8°	6°	5°·4	8°·8	10°·4	15°·2	17°·3	18°	19°·1	17°	13°·2	12°

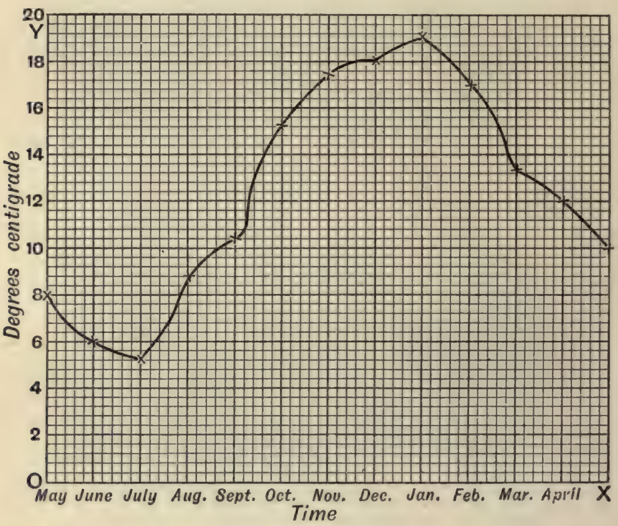


FIG. 22.

Ex. 7. *In the following table are given in hundreds of thousands of pounds sterling the values of the frozen mutton and lamb sent from New Zealand to the United Kingdom for the years named: exhibit the results graphically.*

Year	1896	'97	'98	'99	'00	'01	'02	'03	'04	'05
Mutton	14·25	16·8	16·5	20·1	18·6	18·4	18·8	24·25	19·8	17·5
Lamb	7·9	10·2	11·4	12·4	12·9	14	17·9	21·6	19	19·5

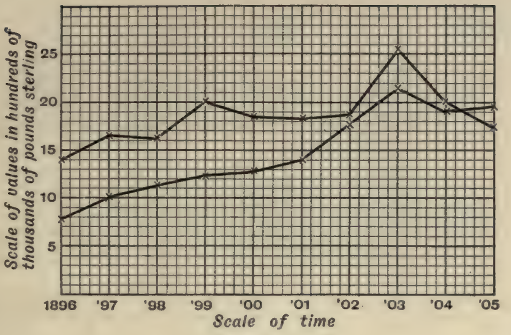


Fig. 23.

EXAMPLES. LXXIII.

1. Given that 7 lbs. of tea cost 8*s.* 9*d.*, construct a graph from which can be read the cost of any number of lbs. between 1 and 30 lbs. and estimate from it the cost of 17 and of 24 lbs. What is the equation of the graph?

2. Draw the graph of $y=3\frac{1}{2}x$, and the graph of $y=2x+3$.

3. The following table of co-ordinates represents a price list of oranges, x denoting the number of oranges and y the price in pence:

x	2	4	5	6	7
y	5	7	8	9	10

(1) Represent these results graphically, and write down the equation of the graph; (2) estimate from the graph the number of oranges that can be got for 1*s.* and the cost of a dozen oranges.

4. The following table represents a price list of caps, x denoting the number of caps and y the cost in shillings:

x	2	3	5	8	etc.
y	5	6·5	9·5	14	etc.

(1) Represent these data by means of a graph, and write down from it the equation of the graph, and (2) estimate from it the cost of 4, 6, 7, and 9 caps.

5. Dredge lifting gravel 50 feet; C = cubic feet excavated per minute; H = horse-power:

H.	10	15	25	50	100
C.	18·2	27·3	45·4	90·9	181·8

Represent these results graphically, and, assuming that the work done is in direct proportion to the power applied, estimate from the graph the number of cubic feet that any dredge of horse-power from 10 to 100 can lift the same height. What is the equation of the graph?

6. Plot on one diagram (pressure on the upper part and rainfall on the lower, and the readings for the same day on the same vertical line) the following readings of the barometer and the rain-gauge for one month. (Note: 1 point = ·01 of an inch).

Pressure	30·5	30·6	30·7	30	29·5	29·3	29·4	29·5	29·5	29·7	30
Rainfall (points)	0	0	0	12	30	50	34	30	28	14	4

Pressure	30·1	30·1	30·4	30·4	30·1	29·5	29·4	29·4	29·6	30·4
Rainfall (points)	2	0	0	0	0	16	40	34	20	8

Pressure	30·5	30·5	30·6	30·4	30·2	30·4	30·4	30·5	30·6	30·4
Rainfall (points)	2	·5	·5	0	0	0	2	0	0	0

What relation between atmospheric pressure and rainfall does the graph seem to indicate?

7. Given that 1 kilometre = ·62 of a mile (approximately), construct a graph from which can be read off any number of kilometres in miles or miles in kilometres. From the graph write down approximately the number of miles in 620 kilometres and the number of kilometres in 560 miles.

8. The following table gives for 1904, to the nearest tenth of a bushel, the yields of wheat and oats per acre in the five Australian States and New Zealand: exhibit these results graphically on the same diagram.

Countries.	Victoria.	N. S. Wales.	Queensland.	South Australia.	West Australia.	Tasmania.	New Zealand.
Wheat	9·5	9·3	14·2	6·5	11·0	18·4	35·4
Oats	18·0	16·0	23·5	11·0	16·3	27·0	42·5

9. Population (in millions) of England and Wales, Scotland, and Ireland for the years named :

Years - - -	1841	1851	1861	1871	1881	1891	1901
England and Wales -	16·0	18·0	20·0	22·7	26·0	29·0	32·5
Scotland - - -	2·6	2·9	3·0	3·4	3·7	4·0	4·5
Ireland - - -	8·2	6·6	5·8	5·4	5·2	4·7	4·5

Represent these results graphically on the same diagram.

CHAPTER XX.

PERCENTAGES.

THE accompanying figure is divided by the heavy lines into 4 equal parts, and by the heavy and light lines together into 100 equal parts. Hence the figure is divided into *fourths* and *hundreds*; and it is seen that $\frac{1}{4}=\frac{25}{100}$.

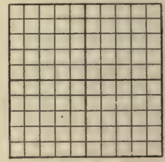


FIG. 24.

In other words, $1:4=25:100$:
also $\frac{1}{2}$ or $\frac{2}{4}$ of the figure $=\frac{50}{100}$ of the figure,
that is, $2:4=50:100$;
and again $\frac{3}{4}$ of the figure $=\frac{75}{100}$ of the figure,
that is, $3:4=75:100$.

Hitherto we have spoken of $\frac{25}{100}$ as 25 hundredths, of $\frac{50}{100}$ as 50 hundredths, and of $\frac{75}{100}$ as 75 hundredths. We shall now name these ratios 25 *per cent.*, 50 *per cent.*, and 75 *per cent.* respectively, the phrase *per cent.* being a contraction for the Latin phrase *per centum*, which means *per hundred.*

A *percentage* is a ratio the consequent of which is 100, and the antecedent of which is called the *rate per cent.*

Hence in the foregoing ratios, 25, 50, and 75 are *rates per cent.*

The figure may be taken to represent any quantity. Verify by a study of it that the following results are true:

100 per cent. of a quantity = the whole of the quantity;
50 per cent. of a quantity $=\frac{1}{2}$ of the quantity;
20 per cent. of a quantity $=\frac{1}{5}$ of the quantity;
25 per cent. of a quantity $=\frac{1}{4}$ of the quantity;
$12\frac{1}{2}$ per cent. of a quantity $=\frac{1}{8}$ of the quantity;
10 per cent. of a quantity $=\frac{1}{10}$ of the quantity;

5 per cent. of a quantity $= \frac{1}{20}$ of the quantity;
$2\frac{1}{2}$ per cent. of a quantity $= \frac{1}{40}$ of the quantity;
4 per cent. of a quantity $= \frac{1}{25}$ of the quantity;
75 per cent. of a quantity $= \frac{3}{4}$ of the quantity;
$33\frac{1}{3}$ per cent. of a quantity $= \frac{1}{3}$ of the quantity.

These results are most useful in mental computations, and therefore they should be remembered.

A rate per cent. being the antecedent of a ratio the consequent of which is 100, it can always be expressed as a fraction. Thus:

30 per cent. of a quantity $= \frac{30}{100} = \frac{3}{10}$ of the quantity;
45 per cent. of a quantity $= \frac{45}{100} = \frac{9}{20}$ of the quantity;
150 per cent. of a quantity $= \frac{150}{100} = 1\frac{1}{2}$ of the quantity;
x per cent. of a quantity $= \dfrac{x}{100}$ of the quantity.

NOTE. The symbol % is frequently used for the phrase *per cent.*

In percentages, the quantity a percentage of which is given or to be found, is regarded as divided into 100 equal parts. Hence

$$\frac{1}{2} \text{ of a quantity} = \frac{1}{2} \text{ of 100 equal parts} = \frac{1 \times 100}{2} = 50 \text{ per cent.,}$$
$$\frac{3}{5} \quad ,, \quad = \frac{3}{5} \text{ of } 100 \quad ,, \quad = \frac{3 \times 100}{5} = 60 \text{ per cent.,}$$
$$\frac{7}{8} \quad ,, \quad = \frac{7}{8} \text{ of } 100 \quad ,, \quad = \frac{7 \times 100}{8} = 87\tfrac{1}{2} \text{ per cent.,}$$
$$\frac{x}{y} \quad ,, \quad = \frac{x}{y} \text{ of } 100 \quad ,, \quad = \frac{x \times 100}{y} \text{ per cent.}$$

Hence, *to express a fraction as a percentage, we multiply its numerator by* 100 *and divide the product by the denominator.*

EXAMPLES LXXIV. (*Oral.*)

Express as percentages:

1. $\frac{5}{8}$, $\frac{4}{9}$, $\frac{9}{20}$, $\frac{3}{8}$, $2\frac{1}{2}$, ·15, ·7, $5\frac{3}{4}$.

2. 1·2, $\frac{5}{16}$, $\frac{1}{8}$, ·75, x, $\dfrac{1}{x}$, $\dfrac{z}{y}$, $4\frac{2}{5}$.

Express as fractions in their lowest terms:

3. 4%, 15%, 80%, 95%, $12\frac{1}{2}\%$, $3\frac{1}{2}\%$.

4. 4·5%, 1·75%, $\frac{1}{2}\%$, $\frac{3}{4}\%$, $\frac{1}{8}\%$, $10x\%$.

Find the values of:

5. 5% of £150, $4\frac{1}{2}\%$ of £75, $12\frac{1}{2}\%$ of £40.

Find what percentage:

6. 8 is of 20, $2\frac{1}{2}$ is of 15, 2*s.* is of £1, 3*s.* 4*d.* is of 10*s.*

7. 4·4 cm. is of a metre, 1 ft. 3 inches is of 3 yds. 6 inches.

Find the quantity of which:

8. £2. 10*s.* is 10%, 3*s.* 6*d.* is 21%, 2 ft. 6 in. is 5%.

9. 2·4 gr. is 12%, 8·9 m. is 10%, £1·6 is 8%.

The following results are useful in mental computation of *trade discount*, which is a percentage reduction of the marked price of goods that are paid for at the time of purchase or shortly after. You will see that 3*d.* is added for every addition of $1\frac{1}{4}$ per cent. to the rate.

$1\frac{1}{4}$ p.c. = 3*d.* in the £.
$2\frac{1}{2}$ p.c. = 6*d.* in the £.
$3\frac{3}{4}$ p.c. = 9*d.* in the £.
5 p.c. = 1*s.* in the £.

$6\frac{1}{4}$ p.c. = 1*s.* 3*d.* in the £.
$7\frac{1}{2}$ p.c. = 1*s.* 6*d.* in the £.
10 p.c. = 2*s.* in the £.
$12\frac{1}{2}$ p.c. = 2*s.* 6*d.* in the £.

Also 1 p.c. = 1 shilling for every £5;
2 p.c. = 2 shillings for every £5; and so on.

And $8\frac{1}{3}$ p.c. = 1*s.* 8*d.* in the £;
$16\frac{2}{3}$ p.c. = 3*s.* 4*d.* in the £;
$33\frac{1}{3}$ p.c. = 6*s.* 8*d.* in the £.

EXAMPLES LXXIV. a.

Use these results to compute the following discounts:

1. $2\frac{1}{2}$ p.c. of £25, of £84, and of £95. 10*s.*

2. $3\frac{3}{4}$ p.c. of £18, 5 p.c. of £50, $6\frac{1}{4}$ p.c. of £27.

3. $7\frac{1}{2}$ p.c. of £45, 10 p.c. of £20. 15*s.*, $12\frac{1}{2}$ p.c. of £29.

4. 3 p.c. of £35, $16\frac{2}{3}$ p.c. of £85, $33\frac{1}{3}$ p.c. of £33. 15*s.*

Ex. 1. *Find* $7\frac{1}{2}$ *per cent of* £17. 16*s*. 6*d*.

By vulgar fractions:

$7\frac{1}{2}$ p.c. of £17. 16*s*. 6*d*

$=\frac{7\frac{1}{2}}{100}$ of £17. 16*s*. 6*d*.

$=\frac{3}{40}$ of £17. 16*s*. 6*d*.

$=\frac{£53.\ 9s.\ 6d.}{40}$

$=£1.\ 6s.\ 8\frac{17}{20}d.$

By decimals (correct to pence):

£17. 16*s*. 6*d*. = £17·825 (to 3 places);

$$\begin{array}{lrr} & \therefore\ 1\text{ p.c.} = & £\cdot17825 \\ & & 7 \\ \hline & \therefore\ 7\text{ p.c.} = & 1\cdot247 \\ \text{and} & \frac{1}{2}\text{ p.c.} = & \cdot089 \\ \hline & \therefore\ 7\frac{1}{2}\text{ p.c.} = & £1\cdot336 \\ & & £1.\ 6s.\ 8\frac{3}{4}d. \end{array}$$

The decimal method is recommended when an exact answer is not required.

EXAMPLES. LXXV.

Find to pence (whole) the value of:

1. 15 per cent. of £95. 12*s*. 6*d*.
2. $3\frac{1}{2}$ per cent. of £125. 19*s*. 5*d*.
3. $8\frac{3}{4}$ per cent. of £735. 16*s*. 11*d*.
4. $2\frac{1}{2}$ per cent. of £437. 14*s*. 5*d*.

Ex. 2. *The population of a village increased from* 80 *to* 100 *in a certain time: find the increase per cent.*

$$\text{Increase} = 100 - 80 = 20;$$

$$\therefore\ \text{increase : original population} = 20 : 80$$
$$= 1 : 4;$$
$$\therefore\ \text{increase} = \tfrac{1}{4}\text{ of the original population};$$
$$\therefore\ \text{increase per cent.} = \frac{1\times100}{4} = 25\text{ per cent.}$$

Ex. 3. *The population of a village decreased from* 100 *to* 72 *in a certain time: find the rate of decrease per cent.*

$$\text{Decrease} = 100 - 72 = 28;$$

$$\therefore\ \text{decrease : original population} = 28 : 100$$
$$= 7 : 25;$$
$$\therefore\ \text{decrease} = \tfrac{7}{25}\text{ of the original population};$$
$$\therefore\ \text{decrease per cent.} = \frac{7\times100}{25} = 28\text{ per cent.}$$

The student should note that in these examples the original population is the unit on which the percentage is to be computed. Hence $\frac{1}{4}$ may be called the *rate of increase per unit*, and $\frac{7}{25}$ the *rate of decrease per unit*.

Such examples are really problems in proportion. In Example 1 we have to find a number that has the same ratio to 100 that 20 has to 80.

Let x represent the number,

$$\text{then}\quad \frac{x}{100}=\frac{20}{80};$$

$$\therefore\ x=\frac{20\times 100}{80}=25;$$

$$\text{for}\quad \frac{25}{100}=\frac{20}{80},\ \text{that is, each ratio}=\frac{1}{4}.$$

Ex. 4. *The population of a town increased* 8 *per cent. between the years* 1900 *and* 1905, *and its population in* 1905 *was* 7425: *find what it was in* 1900.

$$\text{Increase}=\tfrac{8}{100}=\tfrac{2}{25}\ \text{of the population in 1900};$$

$$\therefore\ \text{population in 1905}=1\tfrac{2}{25}\ \text{of the population in 1900}$$

$$=\tfrac{27}{25}\quad ,,\quad ,,\quad ,,\quad ;$$

$$\therefore\ \tfrac{27}{25}\ \text{of the population in 1900}=7425;$$

$$\therefore\ \text{population in 1900}=7425\times\tfrac{25}{27}$$

$$=6875.$$

Or thus:

$$\frac{\text{population in 1900}}{\text{population in 1905}}=\tfrac{25}{27};$$

$$\therefore\ \tfrac{2}{27}\ \text{of the population in 1905}=\text{the increase};$$

$$\therefore\ \text{the increase}=\tfrac{2}{27}\ \text{of}\ 7425=550;$$

$$\therefore\ \text{population in 1900}=7425-550$$

$$=6875.$$

Ex. 5. *A man spends* 87½ *per cent. of his income, and saves* £62. 10*s.*: *find his income.*

Percentages being fractions having a common denominator, namely, 100, we can add, subtract, multiply, or divide them if they are percentages of the same quantity.

The man saves 100 p.c. − 87½ p.c. = 12½ p.c. of his income;

$$\therefore\ 12\tfrac{1}{2}\ \text{p.c. of his income}=\pounds 62.\ 10s.,$$

$$\text{that is,}\quad \frac{12\frac{1}{2}}{100}\ \text{of his income}=\pounds 62.\ 10s.;$$

$$\therefore\ \text{his income}=\pounds 62.\ 10s.\times\frac{100}{12\frac{1}{2}}$$

$$=\pounds 500.$$

Ex. 6. *Determine the percentage of error if* 4 : 5 *be taken for* 2 : 3.

The ratios are $\frac{4}{5}$ and $\frac{2}{3}$.

The L.C.M. of the denominators is 15;

$$\therefore\ \tfrac{4}{5}=\tfrac{12}{15}\ \text{and}\ \tfrac{2}{3}=\tfrac{10}{15};$$

$\therefore$ difference or error $=\frac{2}{15}$, an error of $\frac{2}{15}$ in $\frac{2}{3}$;

$\therefore$ ratio of error to the unit on which it is computed

$$=\frac{2}{15}:\frac{2}{3}=2:10;$$

$\therefore$ percentage of error $=\frac{2\times100}{10}=20$ per cent.

Ex. 7. *Two vessels contain a mixture of wine and water, the first containing* $22\frac{1}{2}$ *per cent. of water and the second* 27 *per cent. of water; from these mixtures another is made of* 5 *parts of the first and* 7 *parts of the second: determine the percentage of water in this mixture.*

Proportion of water in the first $=\frac{22\frac{1}{2}}{100}=\frac{9}{40}$;

$\therefore$ $\frac{9}{40}$ of what is taken from the first is water.

Proportion of water in the second $=\frac{27}{100}$;

$\therefore$ $\frac{27}{100}$ of what is taken from the second is water.

Now, $\frac{5}{12}$ of the new mixture is taken from the first,

and $\frac{7}{12}$,, ,, ,, ,, second;

$\therefore$ water from first + water from second

$$=\left(\tfrac{9}{40}\text{ of }\tfrac{5}{12}+\tfrac{27}{100}\text{ of }\tfrac{7}{12}\right)\text{ of the whole}$$
$$=(\cdot09375+\cdot1575)\text{ of the whole}$$
$$=\cdot25125\text{ of the whole}$$
$$=25\cdot125\text{ or }25\tfrac{1}{8}\text{ per cent.}$$

Ex. 8. *A ship depreciates in value each year at the rate of* 10 *per cent. of its value at the beginning of the year, and its value at the end of* 3 *years is* £14580: *what was its original value?*

At the end of each year the value of the ship is $\frac{9}{10}$ of what it was at the beginning of the year;

$\therefore$ its value at end of 1st year $=\frac{9}{10}$ of its original value,

its value at end of 2nd year $=\frac{9}{10}$ of $\frac{9}{10}$ of its original value,

and its value at end of 3rd year $=\frac{9}{10}$ of $\frac{9}{10}$ of $\frac{9}{10}$ of its original value

$=\frac{729}{1000}$ of its original value;

and its value at the end of the 3rd year = £14580;

$\therefore$ $\frac{729}{1000}$ of its original value = £14580;

$\therefore$ its original value $=£\frac{14580\times1000}{729}=£20000$.

Work the example, putting x for the original value.

Ex. 9. *A man in business loses in his first year* 5 *per cent. of his capital, but in his second year he gains* 6 *per cent. of what he had at the end of the first year, and his capital is now* £14 *more than at the first: find his original capital.*

At beginning of 2nd year capital $= \frac{95}{100}$ of original capital;

$\therefore$ gain during this year $= \frac{6}{100}$ of $\frac{95}{100}$ of original capital

$= \frac{57}{1000}$ of original capital;

$\therefore$ capital + gain $= \left(\frac{95}{100} + \frac{57}{1000}\right)$ of original capital

$= 1\frac{7}{1000}$ of original capital;

$\therefore$ gain $= \frac{7}{1000}$ of original capital;

but £14 is the gain on the original capital;

$\therefore$ $\frac{7}{1000}$ of original capital $=$ £14;

$\therefore$ original capital $= £\frac{14 \times 1000}{7} = £2000$.

Work the example, putting x for the original capital.

EXAMPLES. LXXVI.

1. A school increased in a certain time from 159 to 212: find the rate per cent. of the increase.

2. The population of a village decreased in a certain time from 1916 to 1437: find the rate per cent. of the decrease.

3. In a school there are 371 boys, 233 girls, and 256 infants: what percentage of the whole school is there in each department?

4. The population of a certain country increased 5 per cent. from 1875 to 1885, and its population in the latter year was 40,841,010: find its population in 1875.

5. If out of every 10,000 children born in one year 3540 die in their first year, what percentage of the number survives?

6. 64 children were examined, of whom 56 passed in reading, 60 in writing, and 54 in arithmetic: find the percentage of passes in each of these subjects.

7. 80 scholars were examined, of whom 8 failed in reading, 10 in writing, and 12 in arithmetic: find the percentage of passes in each of these subjects.

8. 650 scholars were examined, of whom 28 per cent. failed in arithmetic, 12 per cent. in reading, and 16 per cent. in writing: find how many children passed in each of these subjects.

9. 90 per cent. of the scholars of a school of 570 pupils are presented for examination: 500 pass in reading, 450 in writing, and 390 in arithmetic: what is the percentage of passes in each of these subjects?

10. A man spends 85 per cent. of his income, and saves £90: find his income.

11. A person paid a tax of 10 per cent. upon his income: what was his income, if, after paying the tax, he had £1251 left?

12. A person spends $\frac{1}{10}$ of his income and then 12 per cent. of the remainder, and has then £5940 left: find his income.

13. A man spends $\frac{1}{25}$ of his money, and then 5 per cent. of what remains: what percentage of his money has he left?

14. A man having 750 sheep sold 8 per cent. of them to one person, 90 of them to another, and $3\frac{1}{2}$ per cent. of the remainder to another: find how many sheep he then had left.

15. If a tax of 12 per cent. on the incomes of a country yields £7,200,000, how much will an income-tax of 5*d.* in the £ produce?

16. Find the annual rent of a farm on which, after a deduction of 20 per cent. has been made, a rate of 3*s.* 1*d.* in the £ on the remainder produces the sum of £38. 17*s.*

17. In graduating his instrument, a cyclometer-maker uses the ratio $\frac{143}{144}$ as equivalent to $\frac{437}{440}$: find to two significant figures the percentage of error.

18. If a man receives 3 per cent. on $\frac{1}{4}$ of his capital, 5 per cent. on $\frac{2}{3}$ of it, and 11 per cent. on the remainder, what percentage does he receive on the whole?

19. If the price of an article be raised 10 per cent., find by how much per cent. a man must reduce his consumption of the article so as not to increase his expenditure.

20. The number of sheep on a farm increases at the rate of 20 per cent. per annum: if at the end of two years there are 720 sheep, how many were there originally?

21. Each of two vessels contains a mixture of brandy and water, the first 15 per cent. of water, and the second 18 per cent.; and from these mixtures another is made of 7 parts of the first and 18 parts of the second: determine the percentage of water in this mixture.

22. A house depreciates in value each year at the rate of 10 per cent. of its value at the beginning of the year, and its value at the end of 3 years is £1093. 10*s.*: find its original value.

23. A person loses in his first year 10 per cent. of his capital, but in the next he gains 20 per cent. of what he had at the end of the first year, and his capital is now £144 more than it was at first: find his original capital.

24. A tradesman's annual losses during 5 years average $1\frac{1}{4}$ per cent. on the capital with which he began, and at the end of the 5 years his effects are worth £2531. 5*s.*: what capital did he begin with?

25. The gross receipts of a railway company in a certain year are apportioned as follows: 41 per cent. to pay the working expenses, 56 per cent. to give the shareholders a dividend at the rate of $3\frac{1}{2}$ per cent. on their shares, and the remainder (£15,000) is reserved: find the company's paid-up capital.

26. A railway company spends £1,616,220 on working expenses during the year, half that sum in laying down new lines, and a third of it in buying rolling stock; the daily receipts are £9238, and at the end of the year a dividend is paid of $6\frac{1}{12}$ on $109\frac{1}{2}$: find the capital of the company.

CHAPTER XXI.

PERCENTAGES: COMMISSION, BROKERAGE, INSURANCE.

An agent is a person who acts for another, and receives for his services, whether in collecting debts or in buying and selling goods for the other, a percentage of the money he collects or of the value of the goods he buys or sells. The money he receives for his services is called his *commission*, and he is sometimes called a *commission agent.*

If the agent buys and sells shares for others, he is generally called a *broker*, and the payment he receives for his services is called *brokerage.*

Insurance is a contract by which for a certain specified payment called a *premium*, a company engages to pay to the person named in the deed of contract a specified sum of money on the occurrence of a specified contingency.

The deed of contract is called the *Policy* or the *Policy of Insurance.*

In *Life Insurance* the contingency is the death of the person named in the policy or his attainment of an age named in the policy; in *Accident Insurance* it is loss or injury by accident: in *Marine Insurance* it is loss of ships, cargoes, etc., at sea; in *Fire Insurance* it is loss by fire.

Companies that undertake such risk are called *Insurance Companies.*

The premium varies with the nature of the risk, and is computed at so much per £100 of the amount insured for. Computations in insurance are therefore computations in percentages.

When a man insures so as to recover the amount specified in the policy and the premium and other expenses connected with the insurance, his insurance is called a *Covered Insurance.*

Heavy insurances are generally undertaken jointly by several parties, each of them writing his name under the policy and undertaking to pay to the insured a specified amount of the sum named in the policy. Such insurers are called *Underwriters*, a name applied most commonly to marine insurers.

The case of "covered insurance" is the only one that needs illustration.

Ex. *A cargo is valued at* £7730; *the premium on insurance is* £3 *per cent., the policy duty is at* 5*s. per cent., and the commission is* $\frac{1}{8}$ *per cent.: what sum must be insured to cover the cargo and the expenses of insurance?*

Premium = £3. 0*s.* 0*d.* per cent.
Duty = 5*s.* 0*d.* per cent.
Commission = 2*s.* 6*d.* per cent.
Expenses = £3. 7*s.* 6*d.* = £$3\frac{3}{8}$ per cent.

Hence it will take $3\frac{3}{8}$ per cent. of the money recovered to pay the total expense of insurance;

∴ the remainder, $96\frac{5}{8}$ per cent. of the money, is the value of the cargo;

$$\therefore \frac{96\frac{5}{8}}{100} \text{ of the money} = £7730;$$

$$\therefore \text{ the money} = £7730 \times \frac{100}{96\frac{5}{8}} = £8000.$$

EXAMPLES. LXXVII.

1. An agent sells goods to the value of £237. 15*s.*, on which he receives a commission of $3\frac{3}{4}$ per cent.: find the amount of his commission.

2. An agent who is paid $2\frac{3}{4}$ per cent. on all money he collects receives £55 as commission: how much has he collected?

3. What annual premium should a person pay to insure his life for £725 at £2. 10*s.* per cent.?

4. A merchant insures a ship and cargo for £15,320 at a premium of $4\frac{1}{4}$ per cent. to cover value of ship and costs of insurance: find the premium paid, and the value of ship and cargo.

5. For what sum should goods worth £4384. 0*s.* 3*d.* be insured at £2. 6*s.* 8*d.* per cent. that the owner may in case of loss recover the value of both goods and premium?

CHAPTER XXII.

PERCENTAGES: PROFIT AND LOSS.

In commercial transactions, profits and losses are generally reckoned as a percentage of the *cost price of the goods* to the producer or the buyer.

Goods sold at more than cost price are sold at a profit, and goods sold at less than cost price are sold at a loss.

In the case of profit, selling price = cost price + profit;
hence selling price − cost price = profit,
and selling price − profit = cost price.

In the case of loss, cost price = selling price + loss;
hence cost price − loss = selling price,
and cost price − selling price = loss.

If an article which cost 2*s.* 6*d.* is sold for 3*s.*, the profit is 6*d.*; for

selling price − cost price = profit.

Hence 6*d.* is the profit on 2*s.* 6*d.* (the cost price);

$$\therefore \text{ rate of profit per unit} = \frac{6d.}{2s.\ 6d.} = \tfrac{1}{5};$$

$$\therefore \text{ rate per cent. of profit} = \frac{1 \times 100}{5} = 20.$$

If an article which cost 2*s.* 6*d.* is sold for 2*s.*, the loss is 6*d.*; for

cost price − selling price = loss.

Hence 6*d.* is the loss on 2*s.* 6*d.* (the cost price);

$$\therefore \text{ rate of loss per unit} = \frac{6d.}{2s.\ 6d.} = \tfrac{1}{5};$$

$$\therefore \text{ rate per cent. of loss} = \frac{1 \times 100}{5} = 20.$$

The actual difference in £. *s*. *d*. between the cost price and the selling price is sometimes called the *absolute profit* or the *absolute loss*, while the percentage of profit or of loss is called the *relative profit* or the *relative loss*; for it is a profit or a loss in relation to the constant 100.

It is important to remember that, when a profit or a loss is expressed as a percentage, it is always as a *percentage of the cost price* unless the contrary is indicated.

If a represents the cost price, and b the gain or the loss, then, a and b being expressed in the same denomination,

$$\text{gain or loss per cent.} = \frac{b}{a} \times 100.$$

If a man sells an article at a profit of 10 per cent., he sells it for $\frac{110}{100}$ of the cost price.

Hence $\frac{110}{100}$ of cost price = selling price.

Ex. 1. *A person buys an article for* £1. 10*s*. ; *what must he sell it for to gain* 15 *per cent.?*

$$\text{Profit} = 15 \text{ per cent.};$$
$$\therefore \text{ selling price} = \tfrac{115}{100} \text{ of cost price}$$
$$= \tfrac{23}{20} \text{ of £1. 10s.}$$
$$= \text{£1. 14s. 6d.}$$

Again, suppose the rate per cent. of profit is 15, and the selling price is £1. 14*s*. 6*d*,

$$\text{then, since } \tfrac{115}{100} \text{ of cost price} = \text{selling price},$$
$$\text{cost price} = \tfrac{100}{115} \text{ of selling price}$$
$$= \tfrac{20}{23} \text{ of £1. 14s. 6d.}$$
$$= \text{£1. 10s.}$$

Hence, if we are given the cost price and the rate of profit, we can find the selling price; and, if given the selling price and the rate of profit, we can find the cost price.

If a man sells an article at a loss of 10 per cent., he sells it for $\frac{90}{100}$ of the cost price.

Hence $\frac{90}{100}$ of cost price = selling price.

Ex. 2. *By selling an article for* 3*s*. 2*d*. *I lose* 5 *per cent.* : *find the cost of the article.*

$$\text{Loss} = 5 \text{ per cent.};$$
$$\therefore \tfrac{95}{100} \text{ of cost price} = 3s.\ 2d.;$$
$$\therefore \text{ cost price} = 3s.\ 2d. \times \tfrac{100}{95}$$
$$= 3s.\ 4d.$$

The following considerations are of great importance:

(*a*) If the gain on an article is 15 per cent., then

$$\text{gain} = \tfrac{15}{100} = \tfrac{3}{20} \text{ of the cost price}$$
$$= \tfrac{3}{20+3} = \tfrac{3}{23} \text{ of the selling price;}$$

for, the gain being $\frac{3}{20}$ of the cost price, the selling price is $(\frac{20}{20}+\frac{3}{20})$ or $\frac{23}{20}$ of the cost price;

$$\text{and } \tfrac{3}{23} \text{ of } \tfrac{23}{20} \text{ of cost price} = \tfrac{3}{20} \text{ of cost price.}$$

And, generally,

$$\text{if gain} = \frac{a}{b} \text{ of the cost price,}$$

$$\text{then gain} = \frac{a}{a+b} \text{ of the selling price.}$$

Conversely,

$$\text{if gain} = \frac{a}{b} \text{ of selling price,}$$

$$\text{then gain} = \frac{a}{b-a} \text{ of the cost price.}$$

(*b*) If the loss on an article is 5 per cent., then

$$\text{loss} = \tfrac{5}{100} = \tfrac{1}{20} \text{ of the cost price}$$
$$= \tfrac{1}{20-1} = \tfrac{1}{19} \text{ of the selling price;}$$

for, after a loss of $\frac{1}{20}$, $\frac{19}{20}$ of the cost is left, that is, the selling price is $\frac{19}{20}$ of the cost price,

$$\text{and } \tfrac{1}{19} \text{ of } \tfrac{19}{20} \text{ of the cost price} = \tfrac{1}{20} \text{ of the cost price.}$$

And, generally,

$$\text{if loss} = \frac{a}{b} \text{ of cost price,}$$

$$\text{then loss} = \frac{a}{b-a} \text{ of selling price.}$$

Conversely,

$$\text{if loss} = \frac{a}{b} \text{ of selling price,}$$

$$\text{then loss} = \frac{a}{b+a} \text{ of the cost price.}$$

Let us apply these important theorems to the solution of the following examples:

Ex. 3. *It is required to find the gain on an article that is sold for £1. 14s. 6d. at a profit of* 15 *per cent.*

$$\begin{aligned}\text{Gain} = 15 \text{ p.c.} &= \tfrac{3}{20} \text{ of cost price}\\ &= \tfrac{3}{23} \text{ of selling price}\\ &= \tfrac{3}{23} \text{ of £1. 14}s. \text{ 6}d.\\ &= 4s.\ 6d.\end{aligned}$$

Ex. 4. *A draper buys an article for* £3. 3s., *and sells so as to gain* 16 *per cent. of the selling price: what does he sell it for?*

$$\begin{aligned}\text{Gain of 16 p.c. of selling price} &= \tfrac{4}{25} \text{ of selling price}\\ &= \tfrac{4}{25-4} \text{ of cost price}\\ &= \tfrac{4}{21} \text{ of £3. 3}s.\\ &= 12s.;\end{aligned}$$

$$\therefore \text{ selling price} = \text{£3. 3}s. + 12s. = \text{£3. 15}s.$$

Or thus directly:

$$\begin{aligned}\text{gain of 16 p.c. of selling price} &= \tfrac{4}{25} \text{ of selling price}\\ &= \tfrac{4}{21} \text{ of cost price};\\ \therefore \text{ selling price} &= 1\tfrac{4}{21} \text{ of cost price}\\ &= \tfrac{25}{21} \text{ of £3. 3}s.\\ &= \text{£3. 15}s.\end{aligned}$$

Ex. 5. *By selling an article for* 3s. 2d. *I lose* 5 *per cent.: find the amount of the loss.*

$$\begin{aligned}\text{Loss} &= \tfrac{1}{20} \text{ of the cost price}\\ &= \tfrac{1}{20-1} \text{ of the selling price}\\ &= \tfrac{1}{19} \text{ of 3}s.\ 2d.\\ &= 2d.\end{aligned}$$

When a tradesman receives payment at once or within a specified time, he usually allows a certain reduction in the marked or "listed" price of his goods. This reduction is called *discount for cash*, and is usually computed at *so much per cent.* on the marked or selling price of the goods. Hence in listing or marking his goods, the tradesman, to guard against loss, adds to the cost price of his goods such a percentage as will enable him to allow discount to his customers without incurring loss to himself.

Suppose, for example, that he wishes to allow a discount of 15 per cent. on the marked price of an article that cost him £8. 10s, and ensure for himself a profit of 10 per cent.,

$$\text{then selling price to gain 10 p.c.} = \tfrac{110}{100} \text{ of cost price,}$$

and this is the price the customer has to pay for the article.

He is allowed 15 p.c. discount on the marked price;

$\therefore$ he pays $\frac{85}{100}$ of the marked price of the article,

and he pays $\frac{110}{100}$ of the cost price of the article;

$$\therefore\ \tfrac{85}{100} \text{ of marked price} = \tfrac{110}{100} \text{ of £8. 10s.};$$

$$\begin{aligned}\therefore\ \text{marked price} &= \tfrac{100}{85} \text{ of } \tfrac{110}{100} \text{ of £8. 10s.}\\ &= \tfrac{22}{17} \text{ of £8. 10s.}\\ &= \text{£11.}\end{aligned}$$

Or thus:

$$\frac{\text{marked price}}{\text{cost price}} = \frac{100 + \text{given p.c. of profit}}{100 - \text{given p.c. of discount}} = \frac{110}{85};$$

$$\begin{aligned}\therefore\ \text{marked price} &= \text{cost price} \times \tfrac{110}{85}\\ &= \text{£8. 10s.} \times \tfrac{110}{85}\\ &= \text{£11, as before.}\end{aligned}$$

Hence, *to mark goods* so as to ensure a given per cent. of profit after allowing a given discount:

(*a*) Find the selling price that will yield the given per cent. of profit and multiply this result by

$$\frac{100}{100 - \text{given p.c. of discount}}.$$

Or (*b*) Multiply the cost price by

$$\frac{100 + \text{given p.c. of profit}}{100 - \text{given p.c. of discount}}.$$

Ex. 6. *By selling an article for* 12*s. I lose* 4 *per cent.: at what price must I sell it to gain* 4 *per cent.?*

$$\begin{aligned}\text{Loss} &= 4 \text{ p.c.};\\ \therefore\ \text{cost price} &= \tfrac{100}{96} \text{ of 12s.};\\ \therefore\ \text{selling price at a gain of 4 p.c.} &= \tfrac{104}{100} \text{ of } \tfrac{100}{96} \text{ of 12s.}\\ &= \text{13s.}\end{aligned}$$

Or thus:

$$\begin{aligned}\text{Loss} &= \tfrac{1}{25} \text{ of cost price}\\ &= \tfrac{1}{24} \text{ of selling price}\\ &= \tfrac{1}{24} \text{ of 12s.} = 6d.;\end{aligned}$$

$$\therefore\ \text{cost price} = 12s. + 6d. = 12s.\ 6d.;$$

$$\therefore\ \text{selling price to gain 4 p.c.} = 12s.\ 6d. \times \tfrac{104}{100} = 13s.$$

Or thus, by proportion:

$$\frac{\text{selling price at a gain of 4 p.c.}}{\text{selling price at a loss of 4 p.c.}}=\frac{104}{96};$$

$$\therefore \text{ selling price at a gain of 4 p.c.}=\frac{104}{96}\text{ of }12s.$$

$$=13s.$$

Ex. 7. *What was the prime cost of an article which, on being sold at a gain of* $7\frac{1}{2}$ *per cent., yields* 2*s.* 1*d. more than if it were sold at a loss of* $12\frac{1}{2}$ *per cent.?*

$$\text{Selling price at a loss}=\frac{87\frac{1}{2}}{100}\text{ of cost price};$$

$$\text{selling price at a gain}=\frac{107\frac{1}{2}}{100}\text{ of cost price};$$

and the difference between these two prices $=2s.\ 1d.$;

$$\therefore \left(\frac{107\frac{1}{2}}{100}-\frac{87\frac{1}{2}}{100}\right)\text{ of cost price}=2s.\ 1d.;$$

$$\therefore \tfrac{1}{5}\text{ of cost price}=2s.\ 1d.;$$

$$\therefore \qquad \text{cost price}=10s.\ 5d.$$

Ex. 8. *An article passes successively through the hands of three dealers, each of whom in selling adds as his profit* 10 *per cent. of the price at which he bought it: if the third dealer sells the article for* £33. 5*s.* 6*d.*, *what did the first dealer pay for it?*

Each dealer gets a profit of $\frac{1}{10}$ of the price he paid;

$\therefore$ the third dealer sells for $(1\frac{1}{10})^3$ of the prime cost of the article;

$$\therefore (\tfrac{11}{10})^3\text{ of the prime cost}=£33.\ 5s.\ 6d.;$$

$$\therefore \tfrac{1331}{1000}\text{ of the prime cost}=£\tfrac{1331}{40};$$

$$\therefore \text{ prime cost}=£25.$$

Ex. 9. *A buys a farm, and sells it to B at a profit of* 5 *per cent., B sells it to C at a profit of* 8 *per cent., and C sells it to D for £x, making a profit of* 15 *per cent.: what did A pay for the farm?*

C gains 15 p.c.;

$$\therefore \text{ cost price to } C=\tfrac{100}{115}\text{ of }£x.$$

B gains 8 p.c.;

$$\therefore \text{ cost price to } B=\tfrac{100}{108}\text{ of }\tfrac{100}{115}\text{ of }£x.$$

A gains 5 p.c.;

$$\therefore \text{ cost price to } A=\tfrac{100}{105}\text{ of }\tfrac{100}{108}\text{ of }\tfrac{100}{115}\text{ of }£x.$$

Ex. 10. *A reduction of* 20 *per cent. in the price of sugar per lb. would enable me to purchase* 12 *lbs. more for* 10 *shillings: find the reduced price per lb.*

Let x represent in pence the present price per lb.,

$$\text{then } \frac{120}{x}=\text{number of lbs. at present price.}$$

The price per lb. at the reduced rate is $\frac{4}{5}x$;

$$\therefore \ \frac{120}{\frac{4}{5}x}=\text{number of lbs. at reduced price;}$$

$$\therefore \ \frac{120}{\frac{4}{5}x}-\frac{120}{x}=12;$$

$$\therefore \ \frac{600}{4x}-\frac{480}{4x}=12;$$

$$\therefore \ 120=48x;$$

$$\therefore \quad x=\tfrac{120}{48}=2\tfrac{1}{2}d.$$

$$\therefore \ \text{reduced price}=\tfrac{4}{5}\text{ of }2\tfrac{1}{2}d.=2d.$$

Ex. 11. *A lb. of tea and* 4 *lbs. of sugar cost* 5 *shillings; but if sugar were to rise* 50 *per cent. and tea* 10 *per cent. they would cost* 6*s.* 2*d.*: *find the cost of the tea and the sugar per lb.*

Let x represent the cost in pence of 1 lb. of tea, and y the cost in pence of 1 lb. of sugar.

Then $1\frac{1}{10}x$ will represent the increased cost of 1 lb. of tea, and $1\frac{1}{2}y$ the increased cost of 1 lb. of sugar.

$$\text{Hence we have} \quad x+4y=60,$$

$$\text{and} \quad 1\tfrac{1}{10}x+6y=74.$$

Clearing of fractions, we have

$$x+4y=60,$$

$$\text{and} \quad 11x+60y=740.$$

Multiply the first equation by 11;

$$\text{then} \quad 11x+44y=660,$$

$$\text{and} \quad \underline{11x+60y=740.}$$

$$\therefore \text{ by subtraction} \quad 16y=80;$$

$$\therefore \ y=5d.$$

Substituting this value for y in the first equation, we have

$$x+20=60;$$

$$\therefore \ x=40d.=3s.\ 4d.;$$

∴ required answer is 5*d.* per lb. for the sugar and 3*s.* 4*d.* per lb. for the tea.

EXAMPLES. LXXVIII.

1. If goods are bought at £2. 5*s*. 10*d*. per cwt. and sold at £2. 11*s*. 4*d*. per cwt., what is the gain per cent.?

2. If tea is bought at 3*s*. 6*d*. per lb. and sold at 3*s*. 10½*d*. per lb., what is the gain per cent.?

3. If eggs are bought at 9*d*. per dozen and sold at 7 for 6*d*., what is the gain per cent.?

4. A man buys articles at £12. 15*s*. per 100, and sells them at 2½ guineas per dozen: find the gain per cent.

5. A piano is bought for £63. 2*s*. 6*d*: what must it be sold for that a gain of 4 per cent. may be made?

6. If I pay £12. 5*s*. for 84 lbs. of tea, what price per lb. must I sell it at to gain 15 per cent.?

7. By selling an article for £46 I gain 15 per cent.: find the cost price of it.

8. A person loses at the rate of 10 per cent. by selling cloth at 15*s*. per yd.: what price per yd. must he charge to gain 20 per cent.?

9. If by selling wine at 15*s*. per gallon I lose 10 per cent., at what price must I sell it to gain 15 per cent.?

10. If 4 per cent. be lost by selling linen at 2*s*. 9*d*. per yard, at what price must it be sold to gain 10 per cent.?

11. If by selling tea at 3*s*. 6*d*. per lb. a grocer gains 12 per cent., how much per cent. would he gain or lose by selling it at 3*s*. per lb.?

12. By selling a horse for £116. 17*s*. a person lost 5 per cent.: what would have been his gain or loss per cent. had he sold him for £132. 4*s*. 6*d*.?

13. If by selling an article for 5*s*. 6*d*. I gain $\frac{3}{8}$ of my outlay, what should I gain per cent. by selling it for 6*s*. 6*d*.?

14. By selling an article for 15 shillings I gain $\frac{7}{8}$ of my outlay: what should I gain per cent. by selling it for 16*s*. 6*d*.?

15. By selling an article for 12*s*. 1½*d*. I lost $\frac{3}{100}$ of the cost of it: what advance should I make on the price of it to gain $\frac{3}{50}$ of the cost of it?

16. By selling goods for £80 I lost $\frac{1}{25}$ of their cost: for what should I have sold them to gain $\frac{3}{20}$ of their cost?

17. A man bought 296 sheep at £1. 7*s*. 6*d*. each, and after paying $\frac{2}{11}$ of his outlay for their keep exchanged them for 37 cows which he sold at £15 per head: what was his gain per cent.?

18. A man mixes 36 gallons of whisky at 15 shillings a gallon with 36 gallons at 13*s*. 6*d*. a gallon, and adds 9 gallons of water: if he sells the mixture at the rate of 19 shillings the gallon, what per cent. does he gain on his outlay?

19. If three kinds of coffee worth 2*s.* 9*d.*, 3*s.* 2½*d.*, and 2*s.* 4½*d.* per lb. respectively be mixed in equal quantities, and the mixture be sold at £16. 9*s.*, per cwt., what is the gain or loss per cent.?

20. A merchant has teas worth 4*s.* 6*d.* and 3*s.* 6*d.* per lb. respectively, which he mixes in the proportion of 3 lbs. of the former to 2 lbs. of the latter, and sells the mixture at 4*s.* 4*d.* per lb.: what does he gain per cent.?

21. A grocer buys coffee at the rate of £8. 10*s.* per cwt., and chicory at £2. 10*s.* per cwt., and mixes them in the proportion of 5 parts of chicory to 7 parts of coffee: at what price must he sell the mixture to gain 16⅔ per cent.?

22. A merchant buys coffee at £5. 12*s.* 6*d.* per cwt., and chicory at £2. 5*s.* 5*d.*, per cwt., and mixes them in the proportion of 5 of the former to 2 of the latter: if he retails the mixture at 1*s.* 3*d.* per lb., what part of the prime cost does he gain?

23. A merchant bought wines at 30*s.*, 40*s.*, and 50*s.* per dozen, which he mixed in the proportions of 5, 4, and 3 respectively: if he sold the mixture at 57*s.* 6*d.* per dozen, what was his gain per cent.?

24. A man buys 100 quills for 4*s.* 7*d.*, and sells them so as to gain 37·5 per cent. of the selling price: find the selling price and the gain per cent.

25. A merchant sells to a retailer at 40 per cent. profit; but, the latter failing, the former receives only 5*s.* in the £: find his gain or loss per cent.

26. A man sold a farm at a profit of 50 per cent.; but the purchaser failed and paid only 10 shillings in the £: how much per cent. did the seller lose?

27. A tradesman's prices are 25 per cent. above cost price: if he allows a customer 10 per cent. on his bill, what profit per cent. does he make?

28. A man sold goods to a customer at a profit of 75 per cent.; but the latter could pay only 75 per cent. of the amount of the bill: how much per cent. did the seller gain by the transaction?

29. A man buys ale at 60 shillings a barrel, and retails it at 1½*d.* per glass of ⅔ of half-a-pint: what percentage does he gain on his outlay?

30. If oranges are bought at 10 a penny, how many should be sold for 2*s.* 2*d.* to gain 30 per cent.?

31. If oranges are bought at 20 for a shilling, how many should be sold for £1. 8*s.* to gain 40 per cent.?

32. Bought eggs at 1*s.* 3*d.* per score: how many may be sold for 2*s.* 3*d.* so as to gain 12½ per cent.?

33. If a trader uses a light weight of 13½ lbs. instead of a stone, how much per cent. does he gain fraudulently?

34. If a grocer uses a light weight of 14½ ozs. for a pound, how much per cent. does he gain fraudulently?

35. By selling tea at 5*s.* 4*d.* per lb. a grocer clears ⅛ of his outlay: if he raises the price to 6*s.* 2*d.*, what does he clear per cent.?

36. A grocer buying goods by means of false scales defrauds to the extent of 15 per cent., and also 15 per cent. in selling: how much per cent. does he gain fraudulently?

37. A person sold 55 yards of silk for £25. 15*s.* 7½*d.*, thereby gaining the cost of 13¾ yards: find the prime cost and the gain per cent.

38. A person sold 20 horses for £175, thereby losing the cost of 6 horses: find the prime cost of the horses, and the loss per cent.

39. An article when sold at a profit of 7½ per cent. yields 2*s.* 1*d.* more than when it is sold at a loss of 12½ per cent.: find the prime cost of it.

40. An article when sold at a gain of 5 per cent. yields 1*s.* 6*d.* more than when sold at a loss of 5 per cent.: find the prime cost of it.

41. The cost of freight and insurance on a certain quantity of goods was 15 per cent., and that of duty 10 per cent. on the original outlay: the goods were sold at a loss of 5 per cent.; but if they had brought £3 more there would have been a gain of 1 per cent.: find how much they cost.

42. I bought note paper at the rate of 3*s.* 7½*d.* for 5 quires, and sold it so as to gain as much on the cost of 32 quires as 3 quires were sold for: at what price did I sell the paper per quire?

43. A bookseller sold a book to a friend at 17 per cent. below cost price: if he had charged 5 shillings more for the book, he would have gained 7 per cent. on his outlay: find the cost price of the book to the bookseller, and the price at which he sold it to his friend.

44. A person by selling an article, which cost £14 per cwt., at 2*s.* 9¾*d.* per lb., makes 5 per cent. more profit than he would do if he sold the whole for £55. 15*s.* 3¾*d.*: what was the quantity sold?

45. A reduction of 10 per cent. in the price of an article would enable me to buy 3 more for £1. 2*s.* 6*d.*: what is the reduced price of the article?

46. A reduction of 40 per cent. in the price of eggs would enable a purchaser to obtain 56 more eggs for £1: find the present price.

47. A reduction of 40 per cent. in the price of half-crown tea enables me to buy for my money 6 lbs. more than I could previously buy for my money: what amount of money have I?

48. A cwt. of sugar when sold at a gain of 5 per cent. yields 2*s.* 3½*d.* more than when sold at a loss of 6 per cent.: find the original cost.

49. A person bought a horse and sold it at a loss of 10 per cent.; if he had received £9 more, he would have gained 12½ per cent.: what did the horse cost him?

50. A draper marks his goods so as to make a profit of 10 per cent. and allow his customers a discount of 10 per cent. on the marked price: what was the cost to him of an article that is listed £1. 2*s.*?

51. The listed price of an article is 18*s.* 8*d.*: if in listing it the seller allowed for a profit of 12 per cent. and a discount of 10 per cent., what did the article cost him?

52. An article cost £1. 18*s.*: at what price should it be marked to allow a profit of 10 per cent. and a discount of 5 per cent. on the marked price?

53. If the manufacturer makes a profit of 20 per cent., the wholesale dealer a profit of 25 per cent., and the shopkeeper a profit of 40 per cent., what was the cost of the manufacture of an article bought at the shop for 17*s.* 6*d.*?

54. *A* sells a piano to *B* at a profit of 5 per cent., *B* sells it to *C* at a profit of 5 per cent., and *C* sells it to *D* at a profit of 5 per cent.: if *D* pays £77. 3*s.* 6*d.* for it. what was its cost to *A*?

55. If 3 per cent. more be gained by selling a horse for £83. 5*s.* than by selling him for £81, what was his original cost?

56. A lb. of tea and 4 lbs. of sugar cost 5 shillings; but, if sugar were to rise 50 per cent. and tea 10 per cent., they would cost 6*s.* 2*d.*: find the cost of the tea and the sugar per lb.

57. 2 lbs. of tea and 5 lbs. of sugar cost 7*s.* 6*d.*; but, if sugar were to rise $33\frac{1}{3}$ per cent. and tea 20 per cent., they would cost 9*s.* 4*d.*: find the cost of the tea and the sugar per lb.

58. 4 horses and 7 cows cost £130; but, if the price of the horses were to rise 25 per cent. and that of the cows 15 per cent., they would cost £155. 10*s.*: find the cost of a horse and of a cow.

59. 12 lbs. of tea and 25 lbs. of coffee together cost £4. 6*s.* 8*d.*; but, if the tea were to rise $2\frac{1}{2}$ per cent. and the coffee to fall $4\frac{1}{2}$ per cent., the same quantities would cost £4. 5*s.* 11*d.*: find the price of the tea and the coffee per lb.

60. A man sells a horse, at a loss, for 40 guineas: had he sold it for 50 guineas his gain would have been $\frac{3}{4}$ of his former loss: find the cost price.

61. A baker's outlay for flour is 70 per cent. of his gross receipts, and other trade expenses amount to $\frac{1}{5}$ of his receipts: the price of flour falls 50 per cent., and the other trade expenses are thereby reduced 25 per cent.: to make the same amount of profit, how much should he now reduce the price of the 5*d.* loaf?

62. A man having bought a quantity of goods for £150, sells $\frac{1}{3}$ at a loss of 4 per cent.: by what increase per cent. must he raise that selling price that by selling the remainder at the increased rate he may gain 4 per cent. on the whole transaction?

CHAPTER XXIII.

MIXTURES.

In the course of trade it is frequently necessary to mix goods of the same kind, but of different qualities, so as to make a mixture worth a particular price. Computing the quantities of the ingredients to be used in making such mixtures does not involve any new principle.

Ex. 1. *A merchant has two kinds of tea, one of which cost him* 2*s.* 6*d. a lb. and the other* 3*s. a lb.*; *he wishes to mix them in such a way as to be able to sell the mixture at* 2*s.* 8*d. a lb.: find in what ratio he must mix them.*

Cost of 1 lb. of one kind = 30*d.*,

cost of 1 lb. of other kind = 36*d.*;

selling price per lb. of the mixture = 32*d.*;

∴ on 1 lb. of the cheaper kind he gains 32*d.* − 30*d.* = 2*d.*,

and on 1 lb. of the dearer kind he loses 36*d.* − 32*d.* = 4*d.*;

∴ he must sell 2 lbs. of the cheaper kind to make good the loss he sustains by the sale of 1 lb. of the dearer kind;

∴ cheaper kind to dearer kind must be as 2 to 1, or some equi-multiple of these numbers.

In all such cases the quantities of the ingredients vary inversely as the deviations of their prices from the mean price.

Ex. 2. *I buy tea at* 2*s.* 3*d. a lb. and other tea at* 3*s.* 6*d. à lb.: in what ratio must I mix them in order that, by selling the mixture at* 4*s. a lb., I may gain* 20 *per cent.?*

Selling price $=\frac{120}{100}$ of mean cost price

$=\frac{6}{5}$ of mean cost price;

∴ mean cost price $=\frac{5}{6}$ of 4*s.* = 40*d.*;

deviation of first price from mean = 40*d.* − 27*d.* = 13*d.* gain;

,, second ,, ,, ,, = 42*d.* − 40*d.* = 2*d.* loss;

∴ quantity of first tea : quantity of second tea = 2 : 13.

Or thus by algebra:

Let x = the number of lbs. of the first kind,
and y = ,, ,, ,, ,, second ,, ;
then $27x+42y=40(x+y)$,
that is, $27x+42y=40x+40y$;

$$\therefore\ 13x=2y;$$

$$\therefore\ \frac{x}{y}=\frac{2}{13},$$

that is, $x:y=2:13$.

Ex. 3. *A merchant mixes two qualities of tea in the proportion of 2 parts of the cheaper to 1 part of the dearer, and, by selling the mixture at 2s. 11d. per lb. he makes a profit of 25 per cent.; on mixing them in the proportion of 3 parts of the cheaper to 2 parts of the dearer and selling the mixture at 3 shillings per lb., he makes the same rate of profit: find the prices at which he bought them.*

Cost price of 1st mixture $=\frac{100}{125}$ of 2s. 11d. = 28d.;
,, ,, 2nd ,, $=\frac{100}{125}$ of 3s. $=\frac{144}{5}$d.

Let x = cost in pence of 1 part of the first mixture,
and y = ,, ,, ,, ,, second ,, ;
then $2x+y$ = cost of 3 parts at 28d. per lb.,
and $3x+2y$ = ,, 5 ,, $\frac{144}{5}$d. per lb.

Hence we have $2x+y=(28\times 3)d.=84d.$

$$3x+2y=\left(\tfrac{144}{5}\times 5\right)d.=144d.$$

$$\therefore\ 6x+3y=252 \text{ (both sides} \times 3\text{)},$$

$$\text{and}\quad 6x+4y=288 \text{ (both sides} \times 2\text{)};$$

$$\therefore\ y=36d.=3s.$$

Substituting this value for y in the first equation,

we have $2x=84d.-36d.=48d.$

$$\therefore\ x=24d.=2s.$$

Ex. 4. *A merchant has three kinds of tea which cost him 2s. 3d., 2s. 7d. and 2s. 10d. per lb. respectively, and he wishes to make from these a mixture worth 2s 6d. per lb.: how is he to mix them?*

Mean price = 30d.;
deviation of price of first from mean = 30d. − 27d. = 3d. gain;
,, ,, second ,, = 31d. − 30d. = 1d. loss;
,, ,, third ,, = 34d. − 30d. = 4d. loss;

We must mix separately each of the ingredients on which a loss is made with that on which a gain is made;

∴ quantity of first : quantity of second = 1 : 3;

and quantity of first : quantity of third = 4 : 3.

The first ingredient occurs in both mixtures, being represented by 1 in the first and by 4 in the second;

∴ 1st kind : 2nd kind : 3rd kind = 5 : 3 : 3.

Ex. 5. *How much coffee at* 1*s*. $7\frac{1}{2}$*d. per lb. and chicory at* $5\frac{3}{4}$*d. per lb. must a person take to make a mixture of* 33 *lbs. worth* 1*s*. $2\frac{1}{2}$*d. per lb.?*

Mean price per lb. of mixture = $14\frac{1}{2}d.$;

deviation of price of coffee from mean = $19\frac{1}{2}d. - 14\frac{1}{2}d. = 5d.$ loss;

,, ,, chicory ,, = $14\frac{1}{2}d. - 5\frac{3}{4}d. = 8\frac{3}{4}d.$ gain;

∴ quantity of coffee : quantity of chicory = $8\frac{3}{4}$: 5

= 35 : 20

= 7 : 4;

∴ quantity of coffee = $\frac{7}{11}$ of 33 lbs. = 21 lbs.,

and quantity of chicory = $\frac{4}{11}$ of 33 lbs. = 12 lbs.

Ex. 6. *A merchant has wines worth* 7*s*., 12*s*., 17*s*. *and* 20*s*. *a gallon respectively, and wishes to make a mixture of them worth* 15*s*. *a gallon: find the proportion in which he should mix them.*

Mean price per gallon = 15*s*.;

deviation of price of first wine = 15*s*. − 7*s*. = 8*s*. gain;

,, ,, second ,, = 15*s*. − 12*s*. = 3*s*. gain;

,, ,, third ,, = 17*s*. − 15*s*. = 2*s*. loss;

,, ,, fourth ,, = 20*s*. − 15*s*. = 5*s*. loss.

We have two gains and two losses; we may combine either gain with either loss:

∴ 1st wine : 3rd wine = 2 : 8,

2nd wine : 4th wine = 5 : 3;

∴ 1st : 2nd : 3rd : 4th = 2 : 5 : 8 : 3.

If we combine the first with the fourth and the second with the third, we have

1st wine : 4th wine = 5 : 8,

2nd wine : 3rd wine = 2 : 3;

then 1st : 2nd : 3rd : 4th = 5 : 2 : 3 : 8.

Ex. 7. *A grocer wishes to mix 24 lbs. of tea at 4s. per lb. with teas worth 6s., 8s., and 9s. per lb. respectively, so as to make a mixture worth 5s. per lb.: what quantities of these must he use?*

Mean price = 5*s*. ;

deviation of price of first from mean = 5*s*. − 4*s*. = 1*s*. gain,
,, ,, second ,, = 6*s*. − 5*s*. = 1*s*. loss,
,, ,, third ,, = 8*s*. 5*s*. = 3*s*. loss,
,, ,, fourth ,, = 9*s*. 5*s*. = 4*s*. loss.

We must combine separately each of the ingredients on which a loss is made with that on which a gain is made;

$$\therefore \quad 1\text{st} : 2\text{nd} = 1 : 1,$$
$$1\text{st} : 3\text{rd} = 3 : 1,$$
$$1\text{st} : 4\text{th} = 4 : 1;$$
$$\therefore \quad 1\text{st} : 2\text{nd} : 3\text{rd} : 4\text{th} = 8 : 1 : 1 : 1;$$

$\therefore$ the first comprises $\frac{8}{11}$ of the mixture;

$\therefore$ $\frac{8}{11}$ of the mixture = 24 lbs.

$\therefore$ weight of mixture = $\left(24 \times \frac{11}{8}\right)$ lbs. = 33 lbs.;

and, obviously, each of the teas that must be used with the given 24 lbs. is $\frac{1}{11}$ of 33 lbs., that is, 3 lbs.

EXAMPLES. LXXIX.

1. In what ratio must sugar worth 3½*d*. per lb. be mixed with sugar worth 5½*d*. per lb. to make a mixture worth 4*d*. per lb.?

2. In what ratio must currants worth 5*d*. per lb. be mixed with currants worth 8*d*. per lb. to make a mixture worth 6*d*. per lb.?

3. In what ratio must tea worth 2*s*. 5*d*. per lb. be mixed with tea worth 3*s*. 4*d*. per lb. to make a mixture worth 2*s*. 9*d*. per lb.?

4. How must a person mix teas worth 3*s*. 1*d*., 2*s*. 9*d*., and 3*s*. 6*d*. per lb. respectively to make a mixture worth 3*s*. per lb.?

5. A merchant has wines worth 8*s*., 10*s*., 14*s*., and 19*s*. a gallon respectively: how must he mix them to make a mixture worth 13*s*. a gallon?

6. A grocer has sugar at 5*d*., 7*d*., 12*d*., and 13*d*. per lb. respectively: how much of each kind will form a mixture worth 10*d*. per lb.?

7. I buy tea at 2*s*. 3*d*. per lb., and other tea at 3*s*. 6*d*.: in what ratio must they be mixed so that selling the mixture at 4*s*. 4*d*. I may gain 30 per cent.?

8. A merchant buys wheat at 39*s*. per quarter, and another kind at 6*s*. per bushel: in what ratio must he mix them to gain 25 per cent. by selling the mixture at £2. 17*s*. 6*d*. per quarter?

9. A grocer buys teas at 2*s.* 8*d.*, 3*s.*, and 3*s.* 9*d.* per lb. respectively: how is he to mix them that by selling the mixture at 4*s.* 1*d.* per lb. he may gain 16⅔ per cent.?

How many lbs. of the 3*s.* 9*d.* tea would be in a mixture containing 3 qrs. 18 lbs. of the 2*s.* 8*d.* tea? Find also how much of each would make 2 cwt. 16 lbs. of the mixture?

10. A druggist makes from ingredients worth 5*s.*, 6*s.*, 8*s.*, and 9*s.* per lb. respectively, 1½ cwt. of a mixture worth 7*s.* per lb.: how much of each ingredient does he use?

11. By mixing two kinds of tea I make a mixture worth 4*s.* 4*d.* per lb.; one kind is worth 5*s.* and the other 3*s.* 6*d.* per lb.: in what ratio are they mixed, and how much of each is there in a mixture of 27 lbs.?

12. I mix wines at 12*s.*, 13*s.*, 14*s.*, and 14*s.* 6*d.* per gallon respectively: if the mixture is worth 13*s.* 6*d.* a gallon, how much of each kind of wine is there in 63 gallons of it?

13. How much tea at 6*s.* per lb. must I mix with 12 lbs. at 3*s.* 8*d.* per lb. to make a mixture worth 4*s.* 4*d.* per lb.?

14. I have 50 lbs. of copper worth 1*s.* 4*d.* per lb., and with this I wish to melt pewter worth 10½*d.* per lb. and brass worth 1*s.* 2*d.* per lb.: find how much brass and pewter I must use to make a mixture worth 1 shilling per lb.

15. I have wines worth 7*s.* 5*d.*, 5*s.* 2*d.*, and 4*s.* 2*d.* per gallon respectively: how much of each of these wines must I mix with 20 gallons worth 6*s.* 8*d.* per gallon to make a mixture worth 6*s.* per gallon?

16. How many lbs. of tea at 2*s.* 8*d.* per lb. must be mixed with 495 lbs. at 2*s.* 9*d.* per lb. so that a profit of 10 per cent. may be made by selling the mixture at 3*s.* per lb.?

17. Strong spirit is mixed with inferior spirit, valued at 5*s.* per gallon, in the ratio of 6 to 1; the mixture is worth 9*s.* a gallon: find the value of a gallon of the strong spirit.

18. I buy two kinds of sugar; if I mix them in the proportion of 3 to 1, the mixture is worth 5½*d.* a lb.; and, if I mix them in the proportion of 5 to 3, the mixture is worth 5¼*d.* a lb.: find the cost per lb. of each kind.

19. A butt of sherry is made up of two wines, A and B, mixed in the ratio of 1 : 3, and the value of the mixture is £70; another butt is made up of the same mixed in the ratio 3 : 1, and the value of the mixture is £50: what is the value of each kind of wine?

20. How much of a substance of specific gravity 2 must be mixed with 20 cubic feet of a substance of specific gravity 10 to make a mixture of specific gravity 5?

CHAPTER XXIV.

PERCENTAGES. SIMPLE INTEREST AND DISCOUNT.

INTEREST is money paid (or received) for the loan of money. It is reckoned at *so much per cent. per annum* on the amount of the loan.

The money borrowed or lent is called the *principal.*

The time for which the money is borrowed or lent may be *a part of a year*, *a year*, or *a number of years.*

If a man borrows £400 for 1 year at 5 per cent., at the end of the year he pays to the lender £20 for the use of the money for that time. If he does not renew the loan, he returns the principal with the interest added to it.

The principal and interest make what is called the *amount.* It is obvious that

$$\begin{aligned} &\text{amount} = \text{principal} + \text{interest}\,; \\ \text{hence}\quad &\text{amount} - \text{interest} = \text{principal}, \\ \text{and}\quad &\text{amount} - \text{principal} = \text{interest}. \end{aligned}$$

If a man puts a sum of money in a bank for a definite time, say, 6 or 12 months, the bank will pay him interest on it. His money is then said to be "at fixed deposit." The "fixed deposit" is in the nature of a loan from him to the bank, which lends it to other customers, charging them a higher rate of interest than it allows to him.

A large amount of money is lent on *mortgage*, and money so lent is generally lent for a number of years, the interest on it being paid quarterly, half-yearly, or yearly, generally half-yearly, during the currency of the mortgage.

A mortgage is a conveyance of property to the lender of money, and is given as security for the return of the money at the end of the time for which the loan is made.

Though loans of the kind we are now considering may extend over a number of years, the interest is paid quarterly, half-yearly, or yearly, and not in a lump sum at the end of two or more years.

If the interest is paid quarterly it is $\frac{1}{4}$, and if it is paid half-yearly it is $\frac{1}{2}$, of the amount of the interest for one year.

Ex. 1. *Find the interest on* £1575 *for* 1 *year at* $3\frac{1}{2}$ *per cent. per annum.*

By vulgar fractions:

$$\begin{aligned} \text{Interest} &= \frac{3\frac{1}{2}}{100} \text{ of £1575} \\ &= \tfrac{7}{200} \text{ of £1575} \\ &= \text{£55. } 2s.\ 6d. \end{aligned}$$

By decimals:

$$\begin{array}{rl} \text{£15·75} & = \text{interest at 1 p.c.} \\ 3\frac{1}{2} & \\ \hline \text{£47·25} & = \text{interest at 3 p.c.} \\ 7·875 & = \text{interest at } \tfrac{1}{2} \text{ p.c.} \\ \hline \text{£55·125} & = \text{interest at } 3\tfrac{1}{2} \text{ p.c.} \\ & = \text{£55. } 2s.\ 6d. \end{array}$$

If the loan is not renewed, the lender receives the sum of the loan and the interest, which is technically called the *amount*;

$$\begin{aligned} \therefore \text{ Amount} &= \text{£1575} + \text{£55. } 2s.\ 6d. \\ &= \text{£1630 } 2s.\ 6d. \end{aligned}$$

Suppose that in the foregoing example we were required to find the interest for, say, 3 years.

$$\begin{array}{rrl} \text{Then,} & \text{£55·125} & = \text{interest for 1 year.} \\ & 3 & \\ \hline & \text{£165·375} & = \text{interest for 3 years.} \\ & & = \text{£165. } 7s.\ 6d. \end{array}$$

This answer means that, after the expiration of three years, the lender will have received as interest the sum of £165. 7*s.* 6*d.*, a third of it at the end of the first year, a third of it at the end of the second year, and a third of it at the end of the third year. If the loan is not renewed, the last payment to the lender is the sum of the principal and the interest for the third year.

Ex. 2. *Find, to the nearest penny, the interest on* £126. 17*s.* 8*d.* *for* 7 *months at* 4½ *per cent. per annum.*

By vulgar fractions:

Interest for 1 year

$$=\frac{4\frac{1}{2}}{100}\text{ of £126. 17}s.\text{ 8}d.\text{;}$$

∴ interest for 7 months

$$=\frac{7}{12}\text{ of }\frac{4\frac{1}{2}}{100}\text{ of £126. 17}s.\text{ 8}d.$$

$$=\frac{\text{£126. 17}s.\text{ 8}d.\times 63}{2400}$$

$$=\frac{\text{£126. 17}s.\text{ 8}d.\times 21}{800}$$

$$=\text{£3. 6}s.\ 7\tfrac{73}{200}d.$$

= £3. 6*s.* 7*d.*, correct to the nearest penny.

By decimals:

£126. 17*s.* 8*d.* = £126·883...

7 months = (½ + ⅙ of ½) year.

$$\begin{array}{rl}
\text{£}1\cdot26883\ldots & =\text{int. for 1 year at 1 p.c.}\\
4\frac{1}{2} & \\
\hline
\text{£}5\cdot0753 & \left.\begin{array}{l}=\text{int. at 4 p.c.}\\ =\text{int. at }\frac{1}{2}\text{ p.c.}\end{array}\right\}\text{for 1 year.}\\
\cdot6344 & \\
\hline
\text{£}5\cdot7097 & =\text{int. at }4\frac{1}{2}\text{ p.c. for 1 year.}\\
\hline
\text{£}2\cdot8549 & =\text{int. for }\frac{1}{2}\text{ year.}\\
\cdot4758 & =\text{int. for 1 month.}\\
\hline
\text{£}3\cdot3307 & =\text{int. for 7 months.}\\
 & =\text{£3. 6}s.\text{ 7}d.
\end{array}$$

Note.—In interest examples it is always sufficient to decimalise the principal to three places.

Ex. 3. *Find (omitting fractions of a penny) the interest on* £247. 15*s.* 9*d.* *from April* 4*th to July* 9*th at* 3½ *per cent. per annum.*

In the computation of interest for days, the first or the last day is omitted. We shall omit the first.

$$\left.\begin{array}{lll}
\text{Days in April} & (30-4) & =26\\
\text{,,}\quad\text{May} & & =31\\
\text{,,}\quad\text{June} & & =30\\
\text{,,}\quad\text{July} & & =\ \ 9
\end{array}\right\}\ \therefore\ \text{days for which interest is computed}=96.$$

Hence the interest $=\frac{96}{365}$ of the interest for 1 year.

Principal = £247·7875;

$$\therefore\ \text{interest}=\frac{96}{365}\text{ of }\frac{3\frac{1}{2}}{100}\text{ of £}247\cdot7875$$

$$=\frac{96\times 3\frac{1}{2}}{365}\text{ of £}2\cdot477875$$

$$=\frac{672}{730}\text{ of £}2\cdot477875$$

$$=\frac{\text{£}\cdot2477875\times 672}{73}$$

$$=\text{£2. 5}s.\text{ 7}d.$$

The following shows the working of the last step. The multiplier is reversed, and its units' digit placed under the fourth decimal digit:

$$\begin{array}{r}
\text{£}\cdot2477875\\
276\\
\hline
\text{£}148\cdot6725\\
17\cdot3451\\
\cdot4956\\
\hline
73\,)\,\text{£}166\cdot5132\,(\,2\cdot281\\
20\ 5\\
5\ 91\\
73
\end{array}$$

Ans. £2·281 = £2. 5*s.* 7*d.*

Knowledge of the following results is sometimes useful in this connexion: $365 = 73 \times 5$; hence

$$\begin{aligned} 73 \text{ days} \qquad\qquad &= \tfrac{1}{5} \text{ of a year,} \\ 146 \text{ days } (= 73 \times 2) &= \tfrac{2}{5} \text{ of a year,} \\ 219 \text{ days } (= 73 \times 3) &= \tfrac{3}{5} \text{ of a year,} \\ 292 \text{ days } (= 73 \times 4) &= \tfrac{4}{5} \text{ of a year.} \end{aligned}$$

The work of the foregoing examples is sometimes expressed symbolically thus:

$$I = \frac{P \times R \times T}{100},$$

where I denotes the interest, P the principal, R the rate per cent. per annum, and T the time in years.

For logarithmic work the following is a more useful formula:

$$I = Prn, \text{ or } I = P \times r \times n,$$

where P denotes the principal, r *the interest of one pound for one year*, and n the time in years.

$$\text{Now,} \quad \text{since } Prn = I;$$

$$\therefore \quad r = \frac{I}{Pn},$$

$$\text{and} \qquad n = \frac{I}{Pr}.$$

Also, if we denote the amount by A, we have

$$A = P + Prn = P(1 + rn),$$

$$\text{whence} \quad \frac{A}{1 + rn} = P,$$

$$\frac{A}{P} = 1 + rn,$$

$$\frac{A - P}{Pr} = n,$$

$$\text{and} \qquad \frac{A - P}{Pn} = r.$$

Ex. 4. *At what rate per cent. per annum will* £720 *amount to* £744. 6*s.* *in* 9 *months?*

Interest = Amount − Principal = £744. 6*s.* − £720 = £24. 6*s.*;

also interest = $£720 \times r \times \frac{3}{4}$;

$\therefore \; £720 \times r \times \frac{3}{4} = £24\frac{3}{10}$;

$$\therefore\ r=\frac{24\frac{3}{10}}{720\times\frac{3}{4}}=\frac{243\times 4}{7200\times 3};$$

$$\therefore\ \text{rate per cent.}=\frac{243\times 4\times 100}{7200\times 3}=4\tfrac{1}{2}.$$

Prove by substituting $4\frac{1}{2}$ for r in the equation.

Another method:

Interest for 9 months = £24. 6*s.*

∴ ,, ,, 1 year $=\frac{4}{3}$ of £24. 6*s.*

= £32. 8*s.*;

$$\therefore\ \text{rate}=\frac{32\frac{2}{5}}{720};$$

$$\therefore\ \text{rate per cent.}=\frac{162\times 100}{720\times 5}=4\tfrac{1}{2}.$$

Ex. 5. *In what time will* £690 *amount to* £710. 14*s.* *at* $4\frac{1}{2}$ *per cent. per annum?*

Interest = £710. 14*s.* − £690

= £20. 14*s.*;

also interest $=£690\times\frac{4\frac{1}{2}}{100}\times n$;

$$\therefore\ £690\times\frac{4\frac{1}{2}}{100}\times n=£20.\ 14s.;$$

$$\therefore\ n=\frac{20\frac{7}{10}}{690\times\frac{4\frac{1}{2}}{100}}$$

$$=\frac{207\times 200}{690\times 10\times 9}$$

$=\frac{2}{3}$ of a year

= 8 months.

Prove by substituting $\frac{2}{3}$ for n in the equation.

Another method:

Total interest = £20. 14*s.*;

interest for 1 year $=\frac{4\frac{1}{2}}{100}$ of £690

$=£\frac{621}{20}$;

$\therefore$ time $=\dfrac{20\frac{7}{10}}{\frac{621}{20}}=\dfrac{2}{3}$ of a year

= 8 months.

Ex. 6. *What sum will amount to* £705. 7*s.* 6*d.* *in* 9 *months at* 6 *per cent. per annum?*

$$P=\frac{a}{1+rn}$$

$$=\frac{£705\frac{3}{8}}{1+\frac{6}{100}\times\frac{3}{4}}$$

$$=\frac{£705\frac{3}{8}}{1\frac{9}{200}}$$

$$=\frac{£5643\times 200}{8\times 209}$$

= £675.

Or thus:

£705. 7*s.* 6*d.* = principal + interest for 9 months

= principal + $\frac{9}{200}$ of principal;

∴ $1\frac{9}{200}$ of principal = £$705\frac{3}{8}$;

∴ principal = $\frac{200}{209}$ of £$705\frac{3}{8}$

= $\frac{200}{209}$ of $\frac{5643}{8}$

= £675.

That is, £675 is the *present worth* of £705. 7*s.* 6*d.* due 9 months hence at 6 per cent. per annum. The reader will find upon trial that £30. 7*s.* 6*d.*, the difference between these two sums, is the interest on £675 for 9 months at 6 per cent. The interest on present worth is technically called *true discount*; that is, *true discount is interest on present worth.*

We shall return to the question of discount presently.

Though examiners and text-books include in simple interest time, rate, and present worth examples involving time periods of more than a year, it should be clearly understood that such examples do not express the usages of commercial life; for, if money is lent for more than a year, the interest, if not paid at the end of the year, would, under ordinary circumstances, be capitalised, and thus made to earn interest for the following year or part thereof.

EXAMPLES. LXXX.

Find (omitting fractions of a penny) the amount of the simple interest on:

1. £1840 for $2\frac{1}{2}$ years at 5 per cent.

2. £360 for $3\frac{1}{4}$ years at 4 per cent.

3. £1000 for $4\frac{3}{4}$ years at $3\frac{1}{2}$ per cent.

4. £605. 5*s.* for $4\frac{1}{5}$ years at $2\frac{3}{4}$ per cent.

5. £120. 5*s.* for 2 years at $3\frac{1}{2}$ per cent.

6. £1813. 19*s.* for 9 months at $3\frac{1}{4}$ per cent.

7. £671. 19*s.* 6*d.* for 3 months at $4\frac{1}{2}$ per cent.

8. £41. 13*s.* 4*d.* for 8 months at $4\frac{1}{2}$ per cent.

9. £1585. 17*s.* 6*d.* for 6 years 8 months at $4\frac{1}{2}$ per cent.

10. £3050. 10*s.* 6*d.* for 3 years 3 months at 5 per cent.

11. £1505. 2*s.* $1\frac{1}{2}$*d.* for 5 years 6 months at 8 per cent.

12. £281. 12*s.* for 4 years 2 months at $3\frac{1}{8}$ per cent.

13. £1276. 8*s.* 4*d.* for 19 years 6 months at 5 per cent.

14. £377. 13*s.* for 15 months at 5 per cent.

15. £6901. for 292 days at $3\frac{1}{2}$ per cent.

16. £215. 12*s.* 6*d.* for 3 years 73 days at $4\frac{1}{2}$ per cent.

17. £750. 12*s.* 6*d.* from March 10th to August 3rd at $4\frac{1}{2}$ per cent.

18. £106. 13*s.* 4*d.* from June 15th to September 18th at $4\frac{1}{2}$ per cent.

19. £1835. 8*s.* 4*d.* lent on the 11th of March and repaid on the 4th of August following at $3\frac{3}{4}$ per cent.

20. At what rate per cent. per annum will £765 amount to £791. 15*s*. 6*d*. in 1 year?

21. At what rate per cent. per annum will £1720 amount to £1767. 6*s*. in 11 months?

22. At what rate per cent. per annum will the interest on £280 amount to £5. 14*s*. 4*d*. in 7 months?

23. The interest on £15 for a period of 3 months amounted to 9 shillings: find the rate per cent. per annum.

24. If 1 penny per week be paid for the loan of 8*s*. 8*d*., what is the rate per cent. per annum?

25. For the loan of £981 for 3 years 4 months I paid £179. 17*s*. as simple interest: find the rate per cent. per annum.

26. In what time will the simple interest on £225 amount to £31. 10*s*. at $3\frac{1}{2}$ per cent. per annum?

27. I lent on mortgage £450 at 4 per cent. per annum, and the total interest amounted to £72: find the time for which the mortgage extended.

28. £1275 was lent for a certain time at $3\frac{4}{5}$ per cent. per annum; the interest, which was paid half-yearly, amounted to £169. 11*s*. 6*d*.: find how long the money was lent.

29. In what time will £640 amount to £653 at $4\frac{1}{16}$ per cent. per annum?

30. What sum will amount to £235. 9*s*. 3*d*. in 1 year at $3\frac{1}{2}$ per cent. per annum?

31. Find the present worth of £1052. 6*s*. due 8 months hence at $4\frac{3}{4}$ per cent. per annum.

32. A loan is made for $2\frac{1}{2}$ years at 3 per cent. per annum, interest payable half-yearly; at the end of the two and a half years it is found that the sum of the principal and all the half-yearly payments is £609. 3*s*. 4*d*.: find the amount of the principal.

33. Find the present value of and the discount on a bill of £2287. 10*s*. due 5 months hence at 4 per cent. per annum.

The simple interest on any sum of money lent for 1 year at x per cent. per annum is equal to $\frac{x}{100}$ of the sum, from which the following useful equations are easily derived.

Suppose the rate to be 3 per cent. per annum, and the time 1 year,

then $\frac{3}{100}$ of principal = interest,
and principal = $\frac{100}{3}$ of interest;
also principal + interest = amount,
that is, $1\frac{3}{100}$ of principal = amount;

hence principal $= \frac{100}{103}$ of amount,
and interest $= \frac{3}{103}$ of amount.

The principal is the present worth of the amount;

$\therefore$ present worth $= \frac{100}{103}$ of amount;

and interest on present worth is true discount;

$\therefore$ true discount $= \frac{3}{103}$ of amount.

And, generally,

if interest $= \dfrac{a}{b}$ of principal,

then true discount $= \dfrac{a}{b+a}$ of amount,

and $\dfrac{a}{b+a}$ of amount $= \dfrac{a}{b}$ of principal.

Ex. 1. *Find the true discount on* £161. 16*s. due* 3 *months hence at* $4\frac{1}{2}$ *per cent. per annum.*

Interest on present worth for 3 months $= \frac{1}{4}$ of $\dfrac{4\frac{1}{2}}{100}$ of present worth
$= \frac{9}{800}$ of present worth
$= \frac{9}{809}$ of amount
$= \frac{9}{809}$ of £161. 16*s.*
$=$ £1. 16*s.*
$=$ true discount.

We can obtain other useful results from the example and its working.

Discount for 3 months $= \dfrac{1\frac{4}{5}}{161\frac{4}{5}} = \frac{9}{809}$ of amount
$= \dfrac{9}{809-9} = \frac{9}{800}$ of present worth;

$\therefore$ interest on present worth for 1 year $= \frac{9\times 4}{800}$ of present worth;

$\therefore$ rate per cent. $= \frac{9\times 4\times 100}{800} = 4\frac{1}{2}$.

Again, the interest on present worth $=$ £1. 16*s.*;

$\therefore$ present worth $=$ £160;

$\therefore$ interest on present worth for 1 year $= \dfrac{4\frac{1}{2}}{100}$ of £160
$=$ £7. 4*s.*;

hence time $= \dfrac{1\frac{4}{5}}{7\frac{1}{5}}$ of a year $= \frac{1}{4}$ of a year or 3 months.

Finally, we show that for the same time and at the same rate *interest on amount − true discount = interest on true discount.*

Amount = £161·8
£1·618
4½
£6·472
·809
4) £7·281
£1·82025 = int. on amt.
£1·8 = true discount.
£·02025
= difference between interest on amount and true discount.

True discount = £1·8.
£·018
4½
£·072
·009
4) £·081
£·02025
= interest on true discount.

It is an easy inference that *true discount is present worth of interest.*

EXAMPLES. LXXXI.

1. Find the true discount on £126. 5*s.* for 3 months at 4 per cent. per annum.

2. The discount on a sum of money due 9 months hence at 4 per cent. per annum is £6. 15*s.* : find the sum.

3. Find the true discount on £257. 8*s.* 8¼*d.* for 210 days at 4½ per cent. per annum.

4. The interest on a certain sum of money for a certain time is £36, and the true discount on the same sum for the same time is £30 : find the sum.

5. The true discount on £196. 4*s.* 4½*d.*, due 6 months hence, is £7. 10*s.* 11¼*d.* : find the rate per cent. per annum of the discount.

6. A man lends a certain sum of money at 4½ per cent. per annum, and at the end of 9 months he receives for interest and principal £1860. 15*s.* : find the sum lent.

CHAPTER XXV.

BILLS OF EXCHANGE AND BANKER'S DISCOUNT.

IN mercantile transactions debts are frequently discharged by what are called Bills of Exchange, a form of which is given below:

£500. WELLINGTON, *January* 9, 1907.

Three months after date pay to myself or order the sum of five hundred pounds for value received.

HENRY SIMS.

Accepted, payable
at the Bank of
N.Z., Wellington.
JAMES THOMSON.

STAMP.

To Mr. JAMES THOMSON,
Cuba St.,
Wellington.

Mr. Thomson has purchased from Mr. Sims (say) goods to the value of £500. It is inconvenient to him to pay at the time of purchase or shortly after, and it is inconvenient to Mr. Sims to wait three months for payment. Mr. Sims therefore sends to Mr. Thomson a bill such as that given above. Mr. Thomson thereupon writes across it the word "accepted," adds the name of the bank at which it is payable, and his signature, and then returns the bill to Mr. Sims.

In "accepting" the bill, Mr. Thomson has undertaken to pay the amount of it three months after January 9, 1907.

The bill is nominally due on the 9th of April, but it is not legally due till three days later, namely on the 12th of April. The three days allowed in extension of time are called *days of grace*, a survival from times when communication between business men was more difficult than it is at present. In all bills of exchange the word month means "calendar month."

Mr. Sims is called the *drawer* or *payee* of the bill, and Mr. Thomson the *acceptor* or *drawee.*

When the drawer receives the "accepted" bill from the drawee, he has three courses open to him:

(1) He may keep the bill till it matures, that is, becomes legally due, and then present it at the bank, the name of which appears on it, whereupon the banker will, if the drawee's account is good, "honour" the bill, that is, pay the amount specified in it. This amount is called *the face value* of the bill.

(2) He may at any time before the maturity of the bill sell it for cash to a banker or a bill-broker. The right to draw the face value of the bill on April 12 now passes from him to the banker or broker, who pays the drawer, not the face value, but the face value less interest on the face value for the time the bill has yet to run before it matures. In doing this the banker or broker is said to *discount* the bill; but, since he charges interest on the face value, his discount is not true discount; that is, it is not interest on the present value, but interest on the amount of the bill.

Banker's or broker's discount is interest on the face value of the bill for the unexpired time of the bill. Such discount is sometimes called *practical* or *commercial discount* in contradistinction to *true* or *theoretical* discount.

(3) He may, after endorsing the bill, use it in payment of his own debts. The bill then becomes the property of his creditor, who, if the drawee should be unable to "meet" it when it matures, is protected against loss by the endorsement of the drawer. Like a banknote, the bill may thus pass from creditor to creditor, the last of whom gets from the bank its face value at the date of its maturity.

If Mr. Sims adopts the second course and sells the bill to a banker or broker on the 1st of March at a discount of 5 per cent. per annum, then

$$\text{unexpired time} = \begin{cases}\text{March } 30\ (=31-1) \text{ days}\\ \text{April } 12 \text{ days}\end{cases}$$
$$= 42 \text{ days};$$

$$\therefore\ \text{banker's discount} = £500 \times \tfrac{42}{365} \times \tfrac{5}{100}$$
$$= £2.\ 17s.\ 6d. \text{ (to the nearest penny).}$$

Hence the banker (or broker) will pay Mr. Sims £500 − £2. 17*s.* 6*d.*; that is, £497. 2*s.* 6*d.*, and will, when the bill matures, present it at the bank named on it, and receive its face value, namely £500.

Ex. 1. *Find the difference between the true and banker's discount on a bill for* £368. 8*s. drawn Sept. 9th at three months, and discounted on Oct. 5th at* 5 *per cent. per annum.*

$$\text{Unexpired time} = \begin{cases} \text{Oct., } 26 \text{ days} \\ \text{Nov., } 30 \quad ,, \\ \text{Dec., } \underline{12} \quad ,, \end{cases}$$

$$\text{Total} = 68 \text{ days} = \tfrac{68}{365} \text{ of a year.}$$

$$\text{True discount for 1 year} = \tfrac{1}{20} \text{ of present value of bill;}$$

$$\begin{aligned} \therefore\ \text{true discount for } \tfrac{68}{365} \text{ year} &= \tfrac{68}{365} \text{ of } \tfrac{1}{20} \text{ of present value of bill} \\ &= \tfrac{17}{1825} \text{ of present value of bill} \\ &= \tfrac{17}{1842} \text{ of amount of bill} \\ &= \tfrac{17}{1842} \text{ of £368. 8s.} \\ &= \tfrac{17}{1842} \text{ of £}\tfrac{1842}{5} \\ &= \text{£3. 8s.} \end{aligned}$$

$$\begin{aligned} \text{Banker's discount} &= \text{£}368{\cdot}4 \times \tfrac{68}{365} \times \tfrac{5}{100} \\ &= \frac{\text{£}3{\cdot}684 \times 68}{73} \\ &= \text{£3. 8s. 8d. nearly.} \end{aligned}$$

$$\text{Hence difference} = 8d.$$

Obviously, the difference between the true and the banker's discount is gain to the banker.

Ex. 2. *The difference between true and banker's discount on a bill for* 6 *months at* 4 *per cent. per annum is* 10*s.* 6*d.*: *find the amount of the bill.*

$$\begin{aligned} \text{Banker's discount} &= \text{Interest on the bill} \\ &= \left(\tfrac{4}{100} \times \tfrac{1}{2}\right) \text{ of the bill} \\ &= \tfrac{1}{50} \text{ of the bill;} \end{aligned}$$

$$\therefore\ \text{true discount} = \tfrac{1}{51} \text{ of the bill;}$$

$$\therefore\ \left(\tfrac{1}{50} - \tfrac{1}{51}\right) \text{ of the bill} = 10\tfrac{1}{2}s.,$$

$$\text{that is,}\quad \frac{1}{50 \times 51} \text{ of the bill} = 10\tfrac{1}{2}s.$$

$$\begin{aligned} \therefore\ \text{amount of bill} &= \text{£}\frac{10\frac{1}{2} \times 50 \times 51}{20} \\ &= \text{£1338. 15s.} \end{aligned}$$

Ex. 3. *If a banker, by discounting a bill at* 6 *per cent. per annum, gets* $6\frac{1}{4}$ *per cent. per annum on his money, how long has the bill to run?*

$$\frac{6}{100} \text{ of the bill} = \frac{6\frac{1}{4}}{100} \text{ of the present value of the hill,}$$

that is, the bill × 6 = present value × $6\frac{1}{4}$;

$$\therefore \text{ bill : present value of bill} = 6\tfrac{1}{4} : 6$$
$$= 25 : 24;$$

$\because$ the banker gains $\frac{1}{24}$ of what he pays for the bill.

His gain per annum $= \frac{6\frac{1}{4}}{100}$ of what he pays for the bill;

$$\therefore \text{ time} = \frac{\frac{1}{24}}{\frac{6\frac{1}{4}}{100}} \text{ of a year} = \frac{100 \times 4}{24 \times 25} \text{ of a year}$$
$$= \tfrac{2}{3} \text{ of a year}$$
$$= 8 \text{ months.}$$

Or thus, by the formula for time:

$$Prn = I;$$

$$\therefore\ n = \frac{I}{Pr} = \frac{\frac{1}{24}}{\frac{6\frac{1}{4}}{100}} = \frac{2}{3} \text{ year as before.}$$

EXAMPLES. LXXXII.

1. A bill was drawn on March the 13th for £325. 10*s*. at 9 months, and discounted on July the 23rd at $3\frac{3}{4}$ per cent. per annum: find the banker's gain.

2. Find the difference between the true and banker's discount on a bill of £743. 10*s*. drawn on Sept. the 25th, at 10 months, and discounted on March the 15th at 5 per cent. per annum.

3. The interest on a certain sum for a certain time is £7, and the true discount on the same sum for the same time is £5. 9*s*. $4\frac{1}{2}$*d*.: find the sum.

4. The difference between the commercial and the true discount on a bill due 9 months hence at 4 per cent. per annum is 15*s*.: find the amount of the bill.

5. If the discount on a bill due 5 months hence at 6 per cent. per annum is £3. 17*s*. 6*d*., what is the amount of the bill?

6. A broker discounted a bill at 4 per cent., making $4\frac{1}{6}$ per cent. on his money: how long had the bill to run from the time when it was discounted?

7. The true discount on a certain sum of money due 9 months hence is £20, and banker's discount on the same sum for the same time is £20. 15*s*.: find the sum and the rate per cent. per annum.

CHAPTER XXVI.

EQUATION OF PAYMENTS.

WHEN different bills become due at different dates, it is sometimes necessary to determine the date when the sum of the bills may be paid with due regard to the interests of debtor and creditor.

The time to elapse before a debt is due is called *term of credit*, and the time to elapse before the sum of two or more debts is due is called *the equated term of credit*. The process by which the equated term of credit is found is called *equation of payments*. The principle it involves is the principle explained in connexion with partnerships.

Ex. 1. *Find the equated term of the following debts:* £40 *payable in* 6 *months*, £100 *payable in* 8 *months, and* £180 *payable in* 12 *months.*

£40 due 6 mos. hence is regarded as equivalent to £240 due 1 mo. hence,
£100 due 8 mos. ,, ,, ,, £800 ,, ,,
£180 due 12 mos. ,, ,, ,, £2160 ,, ,,
∴ £320 due sometime between 6 and 12 mos. hence = £3200 ,, ,,

$$\therefore \text{ equated term of credit} = \tfrac{3200}{320} \text{ mos.} = 10 \text{ mos.}$$

The following is the usual setting for the solution of such examples:

$$\text{Equated time of credit} = \frac{40 \times 6 + 100 \times 8 + 180 \times 12}{40 + 100 + 180}$$
$$= 10 \text{ months.}$$

Ex. 2. *Find the equated term of the following bills:* £47. 12*s.* 7*d.*, £78. 16*s.* 4*d.*, *and* £105. 15*s.*, *the first having* 35 *days to run, the second* 48 *days, and the third* 27 *days.*

$$\text{Equated term} = \frac{47{\cdot}629 \times 35 + 78{\cdot}816 \times 48 + 105{\cdot}75 \times 27}{47{\cdot}629 + 78{\cdot}816 + 105{\cdot}75} \text{ days}$$
$$= \frac{8305{\cdot}433}{232{\cdot}195} \text{ days}$$
$$= 35{\cdot}7 \text{ days.}$$

That is, the bills may be equitably paid 36 days hence.

The equated term of credit thus obtained is only approximate, for the process by which it is obtained assumes what is not necessarily true, namely, that the interest on bills that are underdue at the equated time is equal to the interest on bills overdue at the equated time.

The most useful application of this process occurs in connexion with what is called *balancing and averaging accounts.*

An *account* is the statement of business transactions between creditor and debtor; a *balance* is the difference between the two sides of an account; and the *averaging* of accounts is the process of finding the date when the balance is due.

Ex. 3. *A merchant supplies goods and receives cash as stated hereunder: find the date on which the balance should be paid to close the account.*

1906. *Dr.* 1906. *Cr.*

		£. s. d.			£. s. d.
March 12	To goods, 3 mos.	175 . 15 . 7½	May 10	By cash	125 . 10 . 9
April 17	,, 4 mos.	75 . 18 . 9	July 2	,,	56 . 15 . 0
May 10	,, 6 mos.	97 . 16 . 6	Aug. 12	,,	45 . 16 . 7

We may reckon the number of days from March 12, the date of the first transaction, or from the 1st of March. The date from which we reckon is called the *zero date.* It is generally the more convenient to take as zero date the 1st of the month in which the first transaction occurred.

Zero date March 1.

Dr.

March 1 to June 12 = 103 days.
March 1 to Aug. 17 = 169 days.
March 1 to Nov. 10 = 254 days.

£175·781 × 103 = £18105·443
£75·937 × 169 = £12833·353
£97·825 × 254 = £24847·550
£349·543 £55786·346
£228·116 £23283·796
Balance £121·427 £32502·550

Cr.

March 1 to May 10 = 70 days.
March 1 to July 2 = 123 days.
March 1 to Aug. 12 = 164 days.

£125·537 × 70 = £ 8787·590
£56·750 × 123 = £ 6980·250
£45·829 × 164 = £ 7515·956
£228·116 £23283·796

$$\therefore \text{ equated period} = \frac{32502{\cdot}550}{121{\cdot}427} \text{ days} = 267{\cdot}6 \text{ or } 268 \text{ days.}$$

The balance £121·427 would therefore be due to the merchant 268 days after March 1, that is, on November 24.

When the balance is on the credit side of the account, that is, in favour of the customer, the equated term extends backwards from the zero date, and the customer receives (or is credited with) the balance together with interest thereon for the equated term.

EXAMPLES. LXXXIII.

1. Find the equated term of payment of £200 due 2 months hence, £250 due 3 months hence, and £300 due 4 months hence.

2. One man owes another £600, of which £200 is payable in 3 months, £150 in 4 months, and the rest in 6 months: find the equated term of credit.

3. Find the equated term of payment of £750, of which half is due in 4 months, three-eighths in 5 months, and the rest in 6 months.

4. Find the equated term of the following bills: £25. 16*s*. 8*d*., £49. 15*s*., and £86. 13*s*. 5*d*., the first having 53 days to run, the second 27 days, and the third 39 days.

5. Find the date on which the balance of the following account should be paid, zero date being April 1:

1906. *Dr.* 1906. *Cr.*

		£. *s.* *d.*			£. *s.* *d.*
April 14	To goods, at 2 mos.	85 . 17 . 6	Ap. 25	By cash	65 . 10 . 0
June 9	,, ,, 3 mos.	78 . 14 . 8	Aug. 18	,, goods	62 . 15 . 6
July 20	,, ,, 4 mos.	67 . 5 . 4	Oct. 12	,, goods	45 . 18 . 6

CHAPTER XXVII.

COMPOUND INTEREST.

As we have seen, in simple interest it is the general rule that, when money is lent for more than a year, it is agreed between borrower and lender that the interest shall be paid at the end of stated periods the maximum length of which is in practice not greater than 1 year.

As lent money is supposed to be always bearing interest, if the borrower, instead of paying the interest at the end of the year or other period, retains it for use, he ought for the next period to pay interest on the sum of the principal and interest, and this he does. The interest is then said to be capitalised, that is, it is made an integral part of the capital on which interest is to be paid at the end of the next period.

When the interest on the principal is not paid at the end of the period, but is added to the principal to form a new principal on which interest is to be computed for another period, the interest is called *compound interest*; and, as this process may be repeated for any number of periods, it is obvious that in compound interest the amount for each period becomes the principal for the succeeding period. Compound interest therefore consists of two parts, namely, (1) *interest on the original principal* and (2) *interest on the interest on the original principal.*

Suppose £100 to be lent for three years at 5 per cent. compound interest reckoned yearly; then the interest for the first year is £5. This the borrower retains, and his loan for the second year becomes £105, the interest on which is £5. 5*s.* This is retained by the borrower, whose loan now becomes £110. 5*s.*, on which the interest for the third year is £5. 10*s.* 3*d.* Hence the total interest for the three years is £15. 15*s.* 3*d.*, of which £15 is interest on the original principal, and 15*s.* 3*d.* interest on the interest. Of the latter, 5*s.* is the interest on the first year's interest (£5), and 10*s.* 3*d.* the interest on the interest (£10. 5*s.*) accrued at the end of the second year and lent for the third year. The student should verify this statement.

The work of finding the compound interest of the example given above may be set out as follows:

I. By vulgar fractions:

Principal = £100

Interest for 1st year $(\frac{1}{20})$ = £5

Amount in 1 year = £105 = $1\frac{1}{20}$ of principal.

Interest for 2nd year $(\frac{1}{20})$ = £5. 5*s.*

Amount in 2 years = £110. 5*s.* = $1\frac{1}{20}$ of $1\frac{1}{20}$ of principal.

Interest for 3rd year $(\frac{1}{20})$ = £5. 10*s.* 3*d.*

Amount in 3 years = £115. 15*s.* 3*d.* = $1\frac{1}{20}$ of $1\frac{1}{20}$ of $1\frac{1}{20}$ of principal.

Or more briefly:

Amount of £100 in 3 years at 5 p.c. compound interest = $(1\frac{1}{20})^3$ of £100, where the index of the power is the number of periods.

Hence the compound interest = £115. 15*s.* 3*d.* − £100 = £15. 15*s.* 3*d.*

Or the interest may be found directly thus:

$(1\frac{1}{20})^3$ of principal = $\frac{21\times21\times21}{20\times20\times20}$ of principal,

that is, amount = $\frac{9261}{8000}$ of principal

= principal + $\frac{1261}{8000}$ of principal;

∴ compound interest = $\frac{1261}{8000}$ of principal

= $\frac{1261}{8000}$ of £100

= £15. 15*s.* 3*d.*

II. By decimals:

Principal	£100
1st year's interest	£5
Amount in 1 year	£105
2nd year's interest	£5·25
Amount in 2 years	£110·25
3rd year's interest	£5·5125
Amount in 3 years	£115·7625
Original principal	£100
Compound interest for 3 years	£15·7625
	= £15. 15*s.* 3*d.*

Mental work:

1st year's interest
= £1·00 × 5 = £5.

2nd year's interest
= £1·05 × 5 = £5·25.

3rd year's interest
= £1·1025 × 5
= £5·5125.

Or thus:

$$\begin{aligned}
5 \text{ per cent. of principal} &= {\cdot}05 \text{ of principal};\\
\therefore \text{ amount at end of 1 year} &= 1{\cdot}05 \text{ of principal};\\
\therefore \text{ amount at end of 3 years} &= (1{\cdot}05)^3 \text{ of principal}\\
&= (1{\cdot}05)^3 \text{ of £}100\\
&= 1{\cdot}157625 \text{ of £}100\\
&= \text{£}115{\cdot}7625;\\
\therefore \text{ compound interest} &= \text{£}115{\cdot}7625 - \text{£}100\\
&= \text{£}15{\cdot}7625 = \text{£}15.\ 15s.\ 3d.
\end{aligned}$$

The interest found directly by the decimal method:

$$\begin{aligned}
\text{Amount at end of 3 years} &= 1{\cdot}157625 \text{ of principal};\\
\therefore \text{ compound interest} &= {\cdot}157625 \text{ of principal}\\
&= {\cdot}157625 \text{ of £}100\\
&= \text{£}15{\cdot}7625 = \text{£}15.\ 15s.\ 3d.
\end{aligned}$$

We shall use the results of the working of our example to show the method of finding the principal when time, rate, and amount are given, and to enable us to formulate two important equations.

In 3 years the amount is $(1\frac{1}{20})^3$ of the principal;

$$\begin{aligned}
\text{hence}\quad (\tfrac{21}{20})^3 \text{ of principal} &= \text{amount};\\
\therefore \text{ principal} &= (\tfrac{20}{21})^3 \text{ of the amount}\\
&= \frac{20 \times 20 \times 20}{21 \times 21 \times 21} \text{ of £}115\ 15s.\ 3d.\\
&= \frac{20 \times 20 \times 20}{21 \times 21 \times 21} \text{ of £}\frac{9261}{80}\\
&= \text{£}100.
\end{aligned}$$

Or, to take the decimal method:

$$\begin{aligned}
(1{\cdot}05)^3 \text{ of principal} &= \text{amount};\\
\therefore \text{ principal} &= \frac{\text{amount}}{(1{\cdot}05)^3}\\
&= \frac{\text{£}115{\cdot}7625}{1{\cdot}157625}\\
&= \text{£}100.
\end{aligned}$$

Putting P for principal, r for $\frac{\text{rate}}{100}$, A for amount, and n for time in periods, we derive from the foregoing work the following equations:

(i) $P(1+r)^n = A$;

(ii) $$P = \frac{A}{(1+r)^n},$$

equations that are adapted to logarithmic computation of compound interest, for from (i) we have

$$A = P(1+r)^n;$$

$$\therefore\ \log A = \log P + n \log(1+r);$$

and from (ii) we have

$$P = \frac{A}{(1+r)^n};$$

$$\therefore\ \log P = \log A - n \log(1+r).$$

Ex. 2. *Find the amount of* £600 *for* 3 *years at* 4 *per cent. per annum compound interest.*

$$P = £600,\ n = 3,\ r = \cdot 04, \text{ and } 1 + r = 1 \cdot 04;$$

$$\therefore\ A = 600\,(1 \cdot 04)^3;$$

$$\begin{aligned} \therefore\ \log A &= \log 600 + 3 \log 1 \cdot 04 \\ &= 2 \cdot 7782 + \cdot 0510 \\ &= 2 \cdot 8292; \end{aligned}$$

$$\text{antilog} = 6748;$$

$$\therefore\ \text{amount} = £674 \cdot 8 \text{ approx.}$$

This result is too small by rather more than ·1, the error being due to the circumstance that the mantissa of the logarithm of each of the factors 600 and 1·04 is inexact in the fourth place.

Ex. 3. *What principal will amount to* £270. 8*s. in* 2 *years at* 4 *per cent. per annum compound interest?*

$$A = £270 \cdot 4,\ \ n = 2,\ \ 1 + r = 1 \cdot 04;$$

$$\therefore\ P = £\frac{270 \cdot 4}{(1 \cdot 04)^2};$$

$$\begin{aligned} \therefore\ \log P &= \log 270 \cdot 4 - 2 \log 1 \cdot 04 \\ &= 2 \cdot 4320 - \cdot 0340 \\ &= 2 \cdot 3980; \end{aligned}$$

$$\text{antilog} = 2500;$$

$$\therefore\ \text{principal} = £250.$$

Though this result is accurate, the student must remember that, in results obtained by four-figure logarithms, the fourth figure is not always trustworthy. Results obtained by division are more trustworthy than those obtained by multiplication; and the reason for this is obvious.

Ex. 4. *Find the amount at compound interest of £325 for 1½ years at 5 per cent., the interest payable half-yearly.*

1½ years = 3 half-yearly periods ;
5 p.c. per year = 2½ p.c. per half-year ;
hence we have 3 periods at 2½ p.c. per period.

By decimals,
correct to 4 places :

$P_1 =$ £325
$I_1 =$ 8·125

$P_2 =$ £333·125
$I_2 =$ 8·3281...

$P_3 =$ £341·4531...
$I_3 =$ 8·5363...

∴ amount = £349·9894...

By logarithms :

$A =$ £325 $(1{\cdot}025)^3$;
∴ $\log A = \log 325 + 3 \log 1{\cdot}025$
$= 2{\cdot}5119 + {\cdot}0321$
$= 2{\cdot}5440$;
antilog = 3499 ;
∴ amount = £349·9 approx.

Approximate results can be obtained by logarithms even when the principal contains shillings and pence.

Ex. 5. *Find the compound interest on* £2743. 17*s.* 6*d.* *for* 3 *years at* 3½ *per cent. per annum, the interest payable yearly.*

£2743. 17*s.* 6*d.* = £2743·875 = £2744 approx. ;
∴ $A =$ £2744 $(1{\cdot}035)^3$;
∴ $\log A = \log 2744 + 3 \log 1{\cdot}035$
$= 3{\cdot}4384 + 0{\cdot}0447$
$= 3{\cdot}4831$;
antilog = 3042 ;
∴ $A =$ £3042 approx. ;
∴ interest = £3042 − £2743·875
= £298·125 = £298. 2*s.* 6*d.*,

a result that is only about 3*s.* 8*d.* in error.

The following examples are to be worked by logarithms, and checked by one of the ordinary methods, the check to be given to the nearest penny.

EXAMPLES. LXXXIV.

1. Find the compound interest of £125 for 3 years at 2½ per cent. per annum.

2. Find the compound interest of £2000 for 3 years at 7½ per cent. per annum.

3. Find the amount of £4800 in 2 years at 4 per cent. per annum compound interest.

4. Find the amount of £500 in 4 years at 5 per cent. per annum compound interest.

5. Find the amount of £875 at the end of 15 months at 4 per cent. per annum compound interest, the interest payable quarterly.

6. Find the principal which in 2 years will amount to £3450 at 5 per cent. per annum.

7. What principal will in 2 years amount to £594·88 at 4 per cent. compound interest?

8. Find the amount of £300 in 2 years at 4 per cent. compound interest, the interest payable half-yearly.

Miscellaneous Examples in Compound Interest.

Ex. 1. *The difference between the simple and the compound interest on a certain sum of money for 3 years at 5 per cent. per annum is* £13. 6*s*. $10\frac{1}{2}$*d*.: *find the sum.*

$$\text{Simple interest} = \tfrac{5\times 3}{100} \text{ or } \tfrac{3}{20} \text{ of the sum.}$$

$$\text{Amount at compound interest} = \left(1\tfrac{1}{20}\right)^3 \text{ of the sum}$$
$$= 1\tfrac{1261}{8000} \text{ of the sum;}$$

$$\therefore \text{ compound interest} = \tfrac{1261}{8000} \text{ of the sum;}$$

and difference between simple and compound interest = £13. 6*s*. $10\frac{1}{2}d$;

$$\therefore \left(\tfrac{1261}{8000} - \tfrac{3}{20}\right) \text{ of the sum} = £13.\ 6s.\ 10\tfrac{1}{2}d;$$

$$\text{that is, } \tfrac{61}{8000} \text{ of the sum} = £13.\ 6s.\ 10\tfrac{1}{2}d;$$

$$\therefore \text{ sum} = \tfrac{8000}{61} \text{ of } £13.\ 6s.\ 10\tfrac{1}{2}d.$$
$$= £1750.$$

Ex. 2. *At what rate per cent. will* £340 *amount to* £367·744 *in* 2 *years at compound interest?*

We have seen that $P(1+r)^n = a$.

Hence $(1+r)^n = \dfrac{a}{P}$;

$$\therefore (1+r) = \sqrt[n]{\frac{a}{P}}$$

$$= \sqrt[2]{\frac{367\cdot744}{340}} \text{ (in the example)}$$

$$= \frac{\sqrt{367744}}{\sqrt{340000}}$$

$$= \frac{\sqrt{10816}}{\sqrt{10000}}$$

$$= \frac{\sqrt{104\times 104}}{\sqrt{100\times 100}}$$

$$= \tfrac{104}{100}$$

$$= 1\tfrac{4}{100};$$

$$\therefore \text{ rate} = 4 \text{ per cent.}$$

Ex. 3. *Find in what time £500 will amount to £578. 16s. 3d. at 5 per cent. compound interest.*

$$P(1+r)^n = A\,; \quad \therefore\ (1+r)^n = \frac{A}{P}\,;$$

$$\therefore\ n\log(1+r) = \log A - \log P\,;$$

$$\therefore\ n = \frac{\log A - \log P}{\log(1+r)},$$

$$\begin{aligned}\text{£}578.\ 16s.\ 3d. &= \text{£}578{\cdot}8125\\ &= \text{£}578{\cdot}8, \text{ correct to 4 figures;}\end{aligned}$$

$$\begin{aligned}\therefore\ n &= \frac{\log 578{\cdot}8 - \log 500}{\log 1{\cdot}05}\\ &= \frac{2{\cdot}7625 - 2{\cdot}6990}{{\cdot}0212}\\ &= \frac{{\cdot}0635}{{\cdot}0212} = 3\,;\end{aligned}$$

$$\therefore\ \text{time} = 3 \text{ years.}$$

Or thus:

$$\text{£}500 \times \left(\frac{21}{20}\right)^n = \text{£}578\tfrac{13}{16} = \text{£}\tfrac{9261}{16}\,;$$

$$\begin{aligned}\therefore\ \left(\frac{21}{20}\right)^n &= \frac{9261}{16 \times 500}\\ &= \frac{9261}{8000}\\ &= \frac{21 \times 21 \times 21}{20 \times 20 \times 20}\\ &= \left(\frac{21}{20}\right)^3;\end{aligned}$$

$$\therefore\ n = 3\,; \text{ that is, time} = 3 \text{ years.}$$

Ex. 4. *Find the number of years in which a sum of money will double itself at 5 per cent. per annum compound interest.*

Let n denote the number of years,

$$\text{then}\quad P\left(\frac{21}{20}\right)^n = 2P\,;$$

$$\therefore\ \left(\frac{21}{20}\right)^n = \frac{2P}{P} = 2\,;$$

$$\therefore\ n(\log 21 - \log 20) = \log 2\,;$$

$$\begin{aligned}\therefore\ n &= \frac{\log 2}{\log 21 - \log 20}\\ &= \frac{{\cdot}3010}{{\cdot}0212}\\ &= 14{\cdot}2 \text{ years.}\end{aligned}$$

EXAMPLES. LXXXV.

1. The difference between the simple and the compound interest of a certain sum lent for 2 years at 4 per cent. is £3. 4*s.*: find the sum.

2. The difference between the simple and the compound interest of a certain sum lent for 2 years at 3¾ per cent. per annum is 7*s.* 3¾*d.*: find the sum.

3. A certain sum amounted in 2 years at 4½ per cent. per annum compound interest to £409. 10*s.* 2¼*d.*, and to £408. 15*s.* in the same time and at the same rate per cent. per annum simple interest: find the sum.

4. In what time will £600 amount to £694. 11*s.* 6*d.* at 5 per cent. per annum compound interest?

5. In what time will £1250 amount to £1352 at 4 per cent. per annum compound interest?

6. At what rate per cent. per annum will £400 amount to £424·36 in 2 years at compound interest?

7. At what rate per cent. per annum compound interest will £166. 13*s.* 4*d.* amount to £176. 16*s.* 4*d.* in 2 years?

8. In what time will a sum of money double itself at 4 per cent. per annum compound interest?

9. In what time will a sum of money treble itself at 3 per cent. per annum compound interest?

10. Find the present value of £100 to be paid 8 years hence, reckoning compound interest at 4 per cent. per annum.

11. The amounts at compound interest of a certain sum at the ends of two successive years were £9. 3*s.* 9*d.* and £9. 12*s.* 11¼*d.*: what was the rate of interest per cent.? If these years were the second and third, what was the sum lent?

12. *A* and *B* invested equal sums, the former investing at 5 per cent. compound interest payable yearly, and the latter at 5½ per cent. simple interest; at the end of 3 years the total received by *B* was £3. 13*s.* 9*d.* more than that received by *A*: what was the sum invested?

13. The compound interest on a certain sum lent for 3 years at 6¼ per cent. per annum is £102. 2*s.* 6*d.*: find the simple interest on the same sum for the same time and at the same rate.

14. A sum of money lent at simple interest for 3 years at 5½ per cent. per annum yields as interest £18. 8*s.* 9*d.* more than it would yield if lent at compound interest for the same time at 5 per cent. per annum: find the sum.

CHAPTER XXVIII.

STOCKS.

Stock is a general name for the nominal value of money invested in companies and in government and other public securities. To the lenders of the money to governments and other public bodies are given certificates (called *bonds* or *debentures*) acknowledging the debt and specifying the amount of interest to be paid at stated intervals on each £100 stock, that is, on each bond or debenture.

Suppose, for example, that a government desires to raise a loan of £1,000,000; it will consider that it has 10,000 bonds to sell, and will invite tenders, for the purchase of them. If the prices tendered be over £100 per bond, the government will get more, but, if under £100, less than £1,000,000 from the sale of the bonds; but, whether a bond be sold for more or for less than £100, it is considered to represent a loan of £100 or £100 stock. If a person buys two such bonds, he is said to have £200 stock, if three bonds, £300 stock, and so on, and he is entitled to a certain periodic payment for every £100 bond he holds.

The holder may dispose of his bonds to anyone for cash; but he cannot recover at will from the government the money he spent in the purchase of them. The currency of such bonds or debentures is for a specified period of years, and the holders of them at the end of the period receive from the government (or other body) that issued them £100 for every bond they hold. During their currency, however, the bonds may have changed hands a good many times. In each change, what the purchaser bought was (1) the right to the interest payable on the bonds while he should hold them, (2) the right to sell his interest in them if he should wish to, and (3) the right to receive their face value if he should be the holder of them when they mature.

Owing to a variety of causes, the cash value of a £100 bond or £100 stock (*i.e.* the amount of cash it would bring if sold in the market) varies considerably from time to time. If its cash value is £100, stock is said to be *at par*; if its cash value is below £100 (say £98), stock is said to be *at a discount*; and, if its cash value is above £100 (say £102), stock is said to be *at a premium*. It is important to remember that the nominal value of stock is constant and its actual or cash value variable. *Hence the value of a bond expressed in terms of stock is invariable, while its value expressed in terms of cash varies from time to time.* If this fact is kept in mind, stock sums present no difficulty.

The terms bond and debenture do not apply to all stocks; not, for example, to British Consolidated Annuities, or Consols, which represent the aggregate of a large number of loans consolidated into one uniform stock, on each £100 unit of which is paid the same annuity, at present £2½. For purposes of computation, however, this unit may be regarded as a thing saleable and purchasable in the same way as bonds and debentures; for, just as in the case of bonds and debentures, the purchaser buys (1) the right to the annuity on each £100 unit of the stock, (2) the right to sell his interest in the stock, and (3) the right to receive the face value of the £100 unit should the government redeem the stock while he is the holder of any part of it.

The following are "quotations" of some colonial stocks on November 30 and December 7, 1906:

	Nov. 30.	Dec. 7.
New South Wales 3½ per cents., - -	99½	100
Victorian 3½ per cents., - - - -	100½	99
South Australian 3½ per cents., - - -	100½	100½
Queensland 3½ per cents., - - - -	100	98½
New Zealand 3½ per cents., - - -	102	100¼
Tasmanian 3½ per cents., - - - -	100	98½
West Australian 3½ per cents., - - -	97½	98

The first column of quotations shows the cash value of a £100-bond of each stock on November 30, and the second its value a week later. In two of the stocks there is shown an advance in cash value during the week, and in four a decline. Each bond of the stocks has a face value of £100, which remains unchanged, however much its cash value or market price may change. The 3½ per cents. gives a name to the stocks and indicates the annual payment on each £100 stock, that is, on each bond or £100 unit of the stock.

The holders of Consols are paid four times a year, and the holders of colonial stocks generally twice a year.

Shares in railway, banking, and other companies, and debentures issued by public bodies are bought and sold in the share market in the same way as Government stocks.

Since each £100-bond represents £100 stock, the number representing any amount of stock divided by 100 gives the number of such bonds represented by that amount of stock; or more briefly:

$$\frac{\textit{stock}}{100} = \textit{number of £100-bonds} \text{ (or £100-units of stock)};$$

and, conversely,

$$\text{£}100\ \textit{stock} \times \textit{number of such bonds}$$
$$= \textit{the amount of stock represented by the bonds.}$$

It is obvious that the cash value of the bonds is the market value of a bond multiplied by the number of bonds; or more briefly:

$$\textit{cash value of bonds} = \textit{market value of bond} \times \textit{number of bonds.}$$

The dividend is so much per £100-bond;

$$\therefore\ \textit{annual income derived from any quantity of stock}$$
$$= \textit{dividend per bond} \times \textit{number of bonds.}$$

The student will see, therefore, that the most important thing to determine in stock sums is the number of bonds or "cents." of stock involved in the transaction. This determined, everything else is easy.

Purchases and sales of stock are conducted by stockbrokers who charge a **brokerage**, usually £$\frac{1}{8}$ per £100 stock, to the buyer and the seller. It is obvious, therefore, that brokerage increases the cost of stock to the buyer and diminishes the amount of money obtained from its sale by the seller. The buyer has in fact to pay £$\frac{1}{8}$ more and the seller receives £$\frac{1}{8}$ less than the price per bond quoted. In working examples in stock, unless it is mentioned, brokerage is to be disregarded.

In the following solutions, we shall use the term bond both for the bond proper and for the £100-unit of stock, or, as it is sometimes called, *cent.* of stock.

Ex. 1. *Find the cash value of and the income derived from* £4350 *Russian* 5 *per cents. at* 108.

$$£4350 \text{ stock} = 43\tfrac{1}{2} \text{ bonds};$$
$$\text{cash value of each bond} = £108;$$
$$\therefore \text{ cash value of the stock} = £108 \times 43\tfrac{1}{2} = £4698.$$

Each bond yields an income of £5;

$$\therefore \text{ income derived from } 43\tfrac{1}{2} \text{ bonds} = £5 \times 43\tfrac{1}{2} = £217.\ 10s.$$

Ex. 2. *How much stock can be purchased in the* 3 *per cents. at* $91\frac{5}{8}$ *for* £2199?

$$\text{Money to be invested} = £2199;$$
$$\text{cash price of each bond} = £91\tfrac{5}{8};$$
$$\therefore \text{ number of bonds} = \frac{2199}{91\frac{5}{8}} = 24;$$
$$\text{and the nominal value of each bond} = £100; \quad \therefore \text{ stock} = £2400.$$

Ex. 3. *What income shall I derive from investing* £17000 *in the New Zealand* $4\frac{1}{2}$ *per cents. when they are at* 2 *per cent. premium?*

$$\text{Money to be invested} = £17000;$$
$$\text{cash value of each bond} = £102;$$
$$\therefore \text{ number of bonds} = \tfrac{17000}{102} = 166\tfrac{2}{3};$$
$$\text{each bond yields } £4\tfrac{1}{2} \text{ interest};$$
$$\therefore \text{ interest on } 166\tfrac{2}{3} \text{ bonds} = £4\tfrac{1}{2} \times 166\tfrac{2}{3} = £750.$$

Ex. 4. *What amount of stock in the* $3\frac{1}{3}$ *per cents. at* 70 *must be bought to produce an annual income of* £550?

$$\text{Amount of interest to be raised} = £550;$$
$$\text{amount of interest on 1 bond} = £3\tfrac{1}{3};$$
$$\therefore \text{ number of bonds} = \frac{550}{3\frac{1}{3}} = 165;$$
$$\therefore \text{ amount of stock} = £16500.$$

Ex. 5. *Find the price of the* $4\frac{1}{2}$ *per cents. when* £8000 *stock can be purchased for* £7350.

$$£8000 \text{ stock} = 80 \text{ bonds};$$
$$\text{cost of 80 bonds} = £7350;$$
$$\therefore \text{ cost of 1 bond} = £\tfrac{7350}{80} = 91\tfrac{7}{8}.$$

Ex. 6. *What must be the price of the* 3 *per cents. that by investing* £32850 *I may receive an income of* £1080 *a year?*

$$\text{Amount of interest to be raised} = £1080;$$
$$\text{interest on 1 bond} = £3;$$
$$\therefore \text{ number of bonds} = \tfrac{1080}{3} = 360;$$
$$\text{amount to be spent in these bonds} = £32850:$$
$$\therefore \text{ price of each bond} = £\tfrac{32850}{360} = £91\tfrac{1}{4}.$$

Ex. 7. *Which stock is the better to invest in, the* $3\frac{1}{2}$ *per cents. at* 93, *or the* 3 *per cents. at* 81?

$$\text{Income in 1st case} = \frac{3\frac{1}{2}}{93} \text{ of the money invested;}$$

$$\text{income in 2nd case} = \frac{3}{81} \text{ of the money invested;}$$

$$\text{and since } \frac{3\frac{1}{2}}{93} \text{ is greater than } \frac{3}{81};$$

$\therefore$ the $3\frac{1}{2}$ per cents. is the better stock to invest in.

The relative values of the fractions $\frac{3\frac{1}{2}}{93}$ and $\frac{3}{81}$ may be represented thus:

$$\frac{567,\ 558}{186 \times 81}.$$

Ex. 8. *A person invests* £4680 *in the* 3 *per cents., and gets an annual dividend therefrom of* £144 : *find the price of stock, brokerage being* $\frac{1}{8}$ *per cent.*

$$\text{Dividend} = £144;$$

$$\text{interest on 1 bond} = £3;$$

$$\therefore \text{ number of bonds purchased} = \frac{144}{3} = 48;$$

$$\text{brokerage on each bond} = £\frac{1}{8};$$

$$\therefore \text{ total brokerage} = £\frac{1}{8} \times 48 = £6;$$

$$\therefore \text{ money spent in the purchase of stock} = £4680 - £6;$$

$$\therefore \text{ price of each bond} = £\frac{4674}{48} = £97\frac{3}{8}.$$

Ex. 9. *At what price must I invest in the* 4 *per cents. so that after paying* 6*d. in the pound income-tax I may receive* $4\frac{1}{2}$ *per cent. on my money?*

I am to get £4 on every bond I buy;

I am to pay a tax of 6*d.* in the £ of my income, that is, 2*s.* on every £4 I receive;

$\therefore$ I receive £4. − 2*s.* or £3·9 on every bond I buy, and this is at the rate of $4\frac{1}{2}$ per cent. on the money I invest in the purchase of a bond;

$$\therefore \frac{4\frac{1}{2}}{100} \text{ of the money I pay per bond} = £3{\cdot}9;$$

$$\therefore \text{ price per bond} = \frac{£3{\cdot}9 \times 100}{4\frac{1}{2}}$$

$$= £86\frac{2}{3}.$$

Ex. 10. *How much* 3 *per cent. stock at par must a man sell in order to purchase enough* 4 *per cent. stock at* $124\frac{7}{8}$ *to produce an income of* £399. 10*s.*, *the usual brokerage being charged on each transaction?*

Cost of each 4 per cent. bond $= £124\frac{7}{8} + £\frac{1}{8} = £125$,

income on each bond $= £4$;

$$\therefore \text{ income} = \frac{4}{125} \text{ of the money to be invested;}$$

$$\text{but income} = £399\frac{1}{2};$$

$\therefore$ $\frac{4}{125}$ of money to be invested $= £399\frac{1}{2}$;

$\therefore$ money to be invested $= £\frac{799 \times 125}{2 \times 4}$;

and this is the sum of money he must realize by the sale of 3 per cent. stock at par.

Now selling price of each 3 per cent. bond $= £100 - £\frac{1}{8} = £99\frac{7}{8}$;

$$\therefore \text{ number of bonds to be sold} = \frac{799 \times 125}{2 \times 4 \times 99\frac{7}{8}} = 125 ;$$

and, since each bond represents £100 stock ;

$\therefore$ 3 per cent. stock to be sold $= £12500$.

Ex. 11. *A person has* £16500 *stock in the* 3 *per cents. and sells out at* $101\frac{1}{8}$, *and invests the proceeds in* 4 *per cent. railway debentures at* $131\frac{7}{8}$: *find the change in his income, a brokerage of* $\frac{1}{8}$ *per cent. being charged on each transaction.*

£16500 stock $= 165$ bonds ;

$\therefore$ 1st income $= £(165 \times 3) = £495$;

selling price of each bond $= £101\frac{1}{8} - £\frac{1}{8} = £101$;

$\therefore$ selling price of 165 bonds $= £(101 \times 165)$;

cost of each railway debenture $= £131\frac{7}{8} + £\frac{1}{8} = £132$;

$\therefore$ number of railway debentures $= \frac{101 \times 165}{132} = 126\frac{1}{4}$;

$\therefore$ 2nd income $= £(126\frac{1}{4} \times 4) = £505$;

$\therefore$ change in income $= £505 - £495 = £10$ gain.

Ex. 12. *A person finds that if he invests his money in the* 4 *per cents. at* 92 *his income will be less by two guineas than if he invests it in the* $4\frac{1}{2}$ *per cents. at par : find the sum to be invested.*

In the 4 per cents. income $= \frac{4}{92}$ of the money to be invested ;

in the $4\frac{1}{2}$ per cents. income $= \frac{4\frac{1}{2}}{100}$ of the money to be invested ;

and the difference between the incomes $= £2\frac{1}{10}$;

$\therefore$ $(\frac{9}{200} - \frac{4}{92})$ of the sum to be invested $= £2\frac{1}{10}$;

$\therefore$ $\frac{7}{4600}$ of the sum to be invested $= £\frac{21}{10}$;

$\therefore$ the sum to be invested $= £\frac{21 \times 4600}{7 \times 10} = £1380$.

Ex. 13. *A person invests* £11900 *partly in Colonial* $4\frac{1}{2}$ *per cent. stock at* $97\frac{1}{2}$, *and partly in Consols* (3 *per cents.*) *at par : if he holds twice as much British as Colonial stock, find the income that he obtains from the whole investment.*

2 British bonds cost £200, 1 Colonial bond costs $£97\frac{1}{2}$;

$\therefore$ cost of 2 British bonds and 1 Colonial $= £297\frac{1}{2}$;

and the person invests £11900 ;

$\therefore$ he purchases $\frac{11900}{297\frac{1}{2}}$, or 40 parcels, each containing 2 British and 1 Colonial ;

∴ number of Colonial bonds = 40,
and number of British bonds = 80;
∴ income arising from Colonial stock = £180,
and income arising from British stock = £240;
∴ total income = £420.

EXAMPLES. LXXXVI.

1. Find the cash value of £11500 stock in the 3 per cents. at $101\frac{3}{8}$, the usual brokerage being charged.

2. Find the cash value of £20700 stock in the 3 per cents. at $96\frac{1}{8}$, the usual brokerage being charged.

3. How much stock in the 4 per cents. at $95\frac{7}{8}$ can be purchased for £5000, the usual brokerage being charged?

4. What amount of 7 per cent. stock at $106\frac{3}{8}$ will £585. 15*s*. purchase, brokerage being $\frac{1}{8}$ per cent., and what income will the investment yield?

5. If a man invests £4065 in $4\frac{1}{2}$ per cent. Dunedin City debentures at $135\frac{3}{8}$, what dividend will he get, the usual brokerage being charged?

6. If I invest £2624 in $4\frac{1}{2}$ per cent. City of Melbourne debentures at $127\frac{7}{8}$, what income shall I derive therefrom, the usual brokerage being charged?

7. What sum of money must be invested in the $3\frac{3}{4}$ per cents. at par to secure an annual income of £250?

8. Find how much money I must invest in the 3 per cent. consols at 10 per cent. below par to obtain an income of £2000 a year.

9. What amount of stock in the $3\frac{1}{4}$ per cents. at $103\frac{7}{8}$ must be bought to produce an annual income of £97. 10*s*.?

10. What amount of stock in the new three per cents. at $92\frac{3}{8}$ must be bought to secure an annual income of £420?

11. Find the price of the $3\frac{1}{2}$ per cents. when £2343. 15*s*. stock can be purchased for £2250, the usual brokerage being charged.

12. Find the price of the 3 per cents. when £2600 stock can be purchased for £2353, brokerage $\frac{1}{8}$ per cent.

13. What must be the price of the $3\frac{1}{2}$ per cents. that by investing £4788 I may obtain an income of £159. 12*s*.?

14. What must be the price of the 4 per cents. that by investing £5580 I may obtain an income of £240, the usual brokerage being charged?

15. Which stock is the better to invest in, the 4 per cents. at 120, or the $2\frac{1}{2}$ per cents. at 75?

16. Which stock is the better to invest in, the 4 per cents. at 102, or the $3\frac{1}{2}$ per cents. at 96?

17. Which is the more profitable, to invest money in the 3 per cents. at $97\frac{1}{2}$, or in 4 per cent. railway debentures at 131, and what would be the difference in income derived from investing £8515 in each?

18. A merchant invests half of a sum of money in the 3 per cents. at 92, and half in the $3\frac{1}{2}$ per cents. at 98: which is the better investment?

19. At what price must I invest in the 4 per cents. to get 4½ per cent. interest on my money?

20. At what price must a person invest in the 3½ per cents. to get 4 per cent. interest on his money?

21. What was the price of consols, when, after paying an income-tax of 5*d.* in the pound, a man received a profit of 3½ per cent. on his money?

22. A man invested in the 3 per cents.: if, after deducting an income-tax of 6*d.* in the pound, he obtained 3½ per cent. interest on the money invested, at what price did he buy?

23. A man derives an income of £108 by investing £3595. 10*s.* in the 3 per cents.: find the price of stock.

24. A man invests £4065 in the 4½ per cent. debentures and gets a dividend therefrom of £135: find the price of the debentures, the usual brokerage being charged.

25. A man had £5000 stock in the 3 per cents., and he sold out at 101½, and invested the proceeds in 5½ per cent. Wellington City debentures at 137½: find the change in his income.

26. A man invested £5850 in the 3 per cents. at 97⅜, and when the stock rose to 99⅞ he sold out: what did he gain, the usual brokerage being charged on each transaction?

27. I bought £2500 Mexican railway stock at 70¼, and on the stock rising to 78½ I sold out: find my gain.

28. A broker who charges ⅛ per cent. commission for each transaction sells for me £2000 from the 3 per cents. at 97⅝, and reinvests the money in railway debentures at 129⅞: what amount of debentures does he buy?

29. If a person invests £1300 in 4 per cent. railway preference shares at 104, what would be his loss if the shares fell 1⅜?

30. How much 3½ per cent. stock at 85⅝ must I sell to purchase enough Victorian 4 per cents. at 109⅜ to produce an income of £769. 10*s.*, the usual brokerage being charged on each transaction?

31. A person finds that, if he invests his money in the 3 per cents. at 95, his income will be greater by £2 than if he invests it in the 3½ per cents. at 114: find the sum to be invested.

32. By investing a certain sum of money in the 3½ per cents. at 72, a man gets £3. 10*s.* less in income than he would get by investing the same sum in the 4½ per cents. at 90: find the sum invested.

33. A certain sum, if invested in the 3 per cents., would give an annual income of £105, and, if invested in railway shares at par, paying 5¼ per cent., would yield £177. 6*s.* 4½*d.*: find the sum and the price of the 3 per cents.

34. A man invests money in the 3 per cents. at 97⅜, and on their rising to 98⅛ he sells out, thereby gaining £24: find how much money he invested.

35. How much per cent. do I obtain for my money by investing in railway stock at 134⅞, paying a dividend of 4½ per cent., the broker's commission being ⅛ per cent.?

36. If I invest my money in shares paying £7 per share when the £100 share is at 122½, I find that I get £35. 10*s.* a year more than if I invest it in 5¼ per cent. bonds at 105: find my capital.

CHAPTER XXIX.

MOTION, VELOCITY, ETC.

VELOCITY is generally defined as *rate of motion.* When a particle traverses equal spaces in equal times, its velocity is said to be *uniform*; and, when it does not traverse equal spaces in equal times, its velocity is said to be *variable.* Trains furnish suitable examples of variable velocity, and the hands of a clock suitable examples of uniform velocity.

Uniform velocity is measured by the space traversed in any given time, as an hour, a minute, a second. A *second* is generally chosen as the *unit* of time, and a *foot* as the *unit* of space. Thus, if a particle traverses s feet in t seconds, the measure of its velocity is expressed by the equation

$$v=\frac{s}{t}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where v denotes velocity, s space or distance traversed, and t the number of units of time in which the distance was traversed.

Equations derived from (1):

$$vt=s, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$t=\frac{s}{v}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

As it is impracticable to measure velocity *at* any instant, it is plain that the equation

$$v=\frac{s}{t}$$

means *average* or *mean* velocity. For example, if a train travelled 60 miles in 2 hours, we have

$$v=\frac{s}{t}=\frac{60}{2}=30 \text{ miles per hour.}$$

It is obvious, however, that within the two hours the train would have travelled sometimes faster and sometimes slower than this. 30 miles per hour is therefore its mean or average velocity in this journey.

Ex. 1. *If a train travels* x *miles in* y *hours, find its velocity* (1) *in miles per hour,* (2) *in yards per minute, and* (3) *in feet per second.*

(1) Velocity in miles per hour:

$$v=\frac{s}{t}=\frac{x}{y}\text{ miles per hour.}$$

(2) Velocity in yards per minute:

$$v=\frac{s}{t}=\frac{x\times1760}{y\times60}=\frac{88x}{3y}\text{ yards per minute.}$$

(3) Velocity in feet per second:

$$v=\frac{s}{t}=\frac{x\times1760\times3}{y\times60\times60}=\frac{22x}{15y}\text{ feet per second.}$$

The relative velocity of two particles moving along the same path is the rate at which the distance between them changes per unit of time. If, for example, A walks x miles and B y miles per hour along the same path in opposite directions, then, whether they approach or recede from each other, their relative velocity is $(x+y)$ miles per hour. If, however, they walk in the same direction, their relative velocity is $(x\sim y)$ miles per hour. That is, when the motion is in opposite directions,

relative velocity = sum of rates;

and, when the motion is in the same direction,

relative velocity = difference of rates.

Ex. 2. *A and B are 35 miles apart; they start at the same time to approach each other, A travelling at the rate of 4 and B at the rate of 3 miles an hour: when and where will they meet?*

Let v denote the sum of their velocities; then

$$t=\frac{s}{v}=\frac{35}{4+3}=\frac{35}{7}=5\text{ hours;}$$

$\therefore$ they will meet in 5 hours, and at a point (4×5) miles, that is 20 miles from A's starting point, and (3×5) miles or 15 miles from B's starting point.

Suppose that they start at the same point and travel in the same direction, and that A gives B 2 hours' start: when and where will A overtake B?

B is 6 miles on his way when A starts; hence, to overtake B, A has to gain 6 miles on him.

Let v denote the difference of their velocities; then

$$t=\frac{s}{v}=\frac{6}{4-3}=6 \text{ hours};$$

$\therefore$ A will overtake B in 6 hours, and at a point 24 miles from the starting point.

Solution by Graphs.

The first diagram represents their motion when it is in opposite directions, and the second when it is in the same direction.

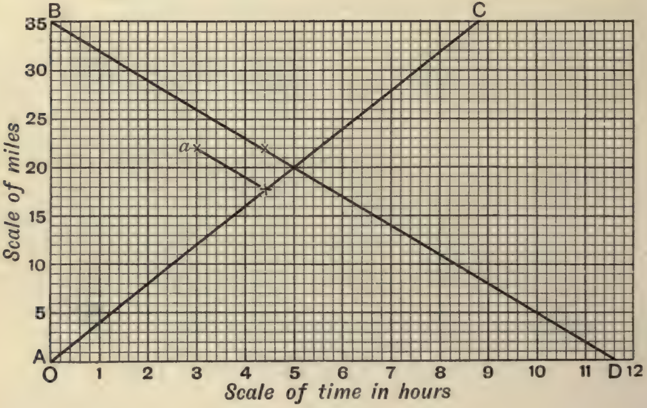


FIG. 25.

From his knowledge of graph notation, the student will readily see that AC is the graph of A's motion and BD the graph of B's; and it is seen from the diagram that in 5 hours the travellers are at the point where the graphs intersect; that is, they meet in 5 hours.

Besides the time when they meet, we can easily read from the diagram the times when they are any given distance apart and the time at which each completes his journey. For example, to find the times when they are 21 miles apart, read on the ordinates 2 and 8 the number of steps between the graphs. The number of steps gives the number of miles the travellers are apart at the times indicated by these ordinates; that is, they are 21 miles apart 2 hours after starting and again 8 hours after starting. To find when they are, say, 4 miles apart, take the point *a* four steps down from a point where the graph of *B*'s motion passes through the corner of a small square, and from this point draw a line parallel to *BD*. Then the point at which this line meets the graph of *A*'s motion is 4 steps down from the graph of *B*'s motion. Hence the travellers are now 4 miles apart, and the time is 4·4 hours after the time when they started. By drawing a similar parallel on the other side of the meeting point, we can find when they are 4 miles apart after passing the meeting point. The diagram shows that *A* completes his journey in $8\frac{3}{4}$ hours, and *B* his in $11\frac{2}{3}$ hours.

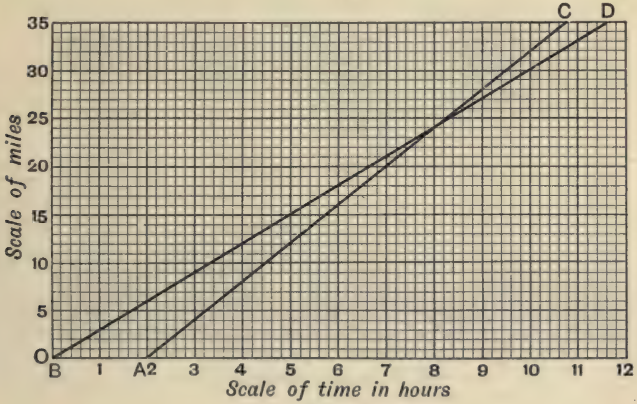


FIG. 26.

The travellers move in the same direction, *A* starting 2 hours after *B*; hence *BD* is the graph of *B*'s motion, and *AC* the graph of *A*'s; *A* therefore overtakes *B* at the point where the graphs intersect, which is on the ordinate indicating 8 hours, that is, 6 hours after *A* started. From the diagram it is easy to read off how far the travellers are apart at any stage of their journey.

Ex. 1. *Two motorists, M and N, set out for a place* 36 *miles distant, M travelling at the rate of* 12 *miles an hour and N at the rate of* 20 *miles an hour; M started at* 2 *p.m. and stopped for half an hour at the end of each hour, and N started at* 4 *p.m. and did the whole journey without stopping: represent the motions graphically, and from the graphs state when and where N passes M, and the times when they arrived at their destination.*

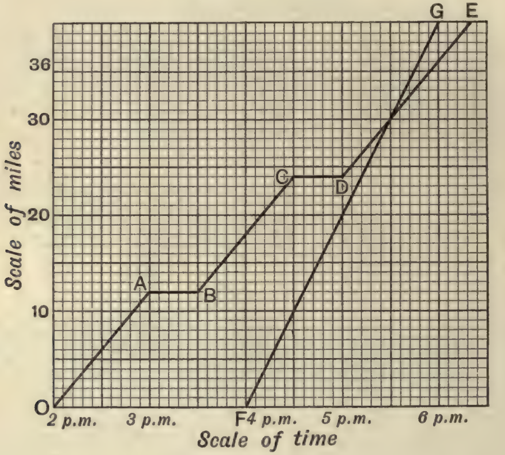


FIG. 27.

The lines OA, BC, and DE are the graphs of M's motion, and the lines AB and CD are the graphs of his rest, that is, graphs representing the movement of time while M was at rest. The line FG is the graph of N's motion, and the point at which it cuts the graph DE indicates the part of the road at which N passed M. The point is seen to be on the abscissa marked 30 miles on the scale of miles and on the ordinate denoting 5·30 p.m. on the time scale; hence N passed M at 5·30 p.m. at a point on the road 30 miles distant from the starting place. M arrived at his destination at 6 p.m., and N at 5·48 p.m.

Ex. 2. *A man travelled from one village to another at the rate of* 16 *miles an hour and returned at the rate of* 8 *miles an hour, and the journey to and from the village occupied* 4 *hours: find the distance between the villages.*

OB is the graph of 16 miles an hour, and CD the graph of 8 miles an hour; and the point at which the graphs intersect indicates the distance of the village, namely $21\frac{1}{3}$ miles from the man's starting point. Read off from the diagram the time taken to go to, and the time taken to return from, the village.

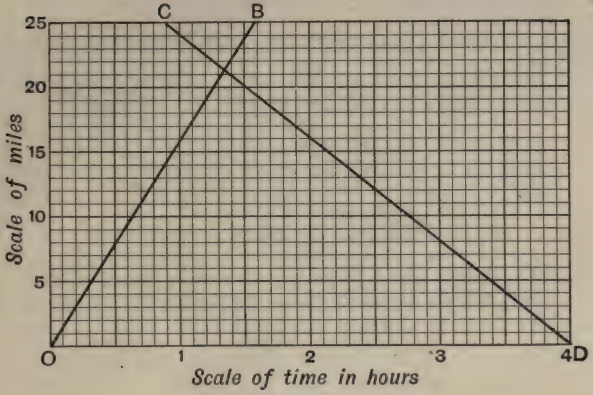


FIG. 28.

Let x denote the distance in miles; then

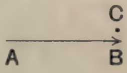
$$\frac{x}{16}+\frac{x}{8}=4,$$

that is, $3x=64$;

$$\therefore\ x=21\tfrac{1}{3}\text{ miles.}$$

Hence the journey to the village occupied $\frac{21\frac{1}{3}}{16}$ hours, and the journey from the village $\frac{21\frac{1}{3}}{8}$ hours.

Let the line AB represent a moving body, the end B of which is in line with the point C; then it is plain that, to pass this point, AB must move through a space equal to its own length.



This is the case of a train or any such body passing an object so small that it may be regarded as a point.

Ex. 3. *How long will it take a train* 88 *yards long, travelling at the rate of* 30 *miles per hour, to pass a certain mile-post?*

The post may be regarded as a point; hence to pass it the train must travel a distance equal to its own length, that is, 88 yards.

$$t=\frac{s}{v}=\frac{88}{30\times 1760}\text{ hours}$$

$$=\frac{88\times 60\times 60}{30\times 1760}\text{ seconds}=6\text{ seconds.}$$

Ex. 4. *A train* 66 *yards long passes a post in* 2 *seconds: what is the rate of the train per hour?*

To pass the post it must travel a distance equal to its own length, that is, 66 yards.

$$\begin{aligned} v=\frac{s}{t}&=\frac{66}{2}\text{ yards per second}\\ &=\frac{66\times 60\times 60}{2}\text{ yards per hour}\\ &=\frac{66\times 60\times 60}{2\times 1760}\text{ miles per hour}\\ &=67\tfrac{1}{2}\text{ miles per hour.}\end{aligned}$$

Suppose the body *AB* has to move past a fixed line *BC*; then it is plain that, when the end *A* of *AB* is in line with the point *C*, the end *B* of *AB* will be beyond the point *C* by the length of *AB*; that is, while *AB* is passing *BC*, the change that takes place in their relative positions is equal to *AB*+*BC*.

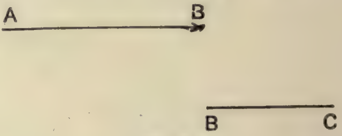


FIG. 29.

This is the case of a train passing through a tunnel, over a bridge; of a column of men passing through a street; and so on.

Ex. 5. *How long will it take a column of men* 180 *feet long to move through a street* ½ *a mile long, at the rate of* 3 *miles an hour?*

From the time the head of the column just enters the street till it reaches the other end, it must travel ½ a mile; then, for the whole column to emerge from the street, it must travel a further distance of 180 feet; that is, the total distance every man in the column has to travel

$$\begin{aligned} &=\text{length of street}+\text{length of column}\\ &=\tfrac{1}{2}\text{ mile}+180\text{ feet}\\ &=(880+60)\text{ yards}=940\text{ yards.}\end{aligned}$$

$$\begin{aligned} t=\frac{s}{v}&=\frac{940}{3\times 1760}\text{ hours}\\ &=\frac{940\times 60}{3\times 1760}\text{ minutes}\\ &=10\tfrac{15}{22}\text{ minutes.}\end{aligned}$$

Ex. 6. *A train running at the rate of* 25 *miles an hour was observed to cross a bridge* $94\frac{4}{9}$ *yards long in exactly* 20 *seconds: find the length of the train.*

From the time the head of the train entered on the bridge till it reached the other end of it, the train travelled $94\frac{4}{9}$ yards; and, to get quite clear of the bridge, it had to travel a further distance equal to its own length;

∴ total distance = length of bridge + length of train;

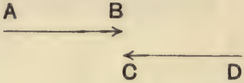
∴ in 20 seconds the train travelled ($94\frac{4}{9}$ yds. + length of train);

but in 20 seconds, at 25 miles an hour, it travelled $\frac{25 \times 1760 \times 20}{60 \times 60}$ yds., that is, $244\frac{4}{9}$ yards;

∴ $94\frac{4}{9}$ yards + length of train = $244\frac{4}{9}$ yards;

∴ length of train = $\left(244\frac{4}{9} - 94\frac{4}{9}\right)$ yards = 150 yards.

Suppose AB and CD to be two trains moving in opposite directions on parallel lines, with the fronts of their engines just in line; then it is obvious that *to pass each other the trains must together move through a space equal to the sum of their lengths.*



This is most easily realised when the trains are of equal lengths and moving with the same velocity; for then to pass each other the train CD must move into a position exactly opposite the present position of the train AB, and the train AB into a position exactly opposite the present position of the train CD; that is, each train must move through a space equal to its own length, and therefore the two must together move through a space equal to the sum of their lengths. *Hence the trains would clear each other in the time it would take a particle having a velocity equal to the sum of their velocities to move a distance equal to the sum of the lengths of the trains.*

It is obvious that, if one of the trains CD were standing still, to pass it the train AB must move a distance equal to the sum of the lengths of the two trains; that is, every point in AB must move through a space equal in length to the sum of the lengths of the two trains. This, too, is the case of a particle having to move a certain distance at a certain velocity.

Ex. 7. *A train* 90 *yards long, going* 40 *miles an hour, meets another train* 86 *yards long, going* 35 *miles an hour: how long do they take to pass each other?*

To pass each other they must together run a distance equal to the sum of their lengths.

Sum of their lengths = (90 + 86) yds = 176 yds. = $\frac{1}{10}$ of a mile;

sum of their velocities = (40 + 35) miles per hour

= 75 miles per hour;

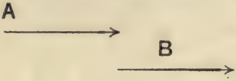
hence we have to find how long it would take a particle moving at the rate of 75 miles an hour to go $\frac{1}{10}$ of a mile.

$$t=\frac{s}{v}=\frac{\frac{1}{10}\text{ mile}}{75\text{ miles}}=\frac{\frac{1}{10}}{75}=\frac{1}{750}\text{ of an hour}$$

$$=\frac{60\times 60}{750}\text{ seconds}=4\tfrac{4}{5}\text{ seconds};$$

and, generally, when two such bodies are moving in opposite directions

$$\text{Time taken to pass each other}=\frac{\text{sum of lengths of moving bodies}}{\text{sum of velocities of the bodies}}.$$



Suppose A and B to be two trains running in the same direction on parallel lines, A running faster than B; and suppose the front of the engine of A to be just in line with the end of the van of B; then it is plain that, to pass B, A must *gain* a distance equal to its own length and that of B; that is, a distance equal to the sum of the lengths of the two trains.

Ex. 8. *If in Ex.* 7 *the trains were moving in the same direction, how long, after getting the front of its engine in line with the end of the van of the slower train, would the faster train take to pass the slower train?*

To pass the slower, the faster train must *gain* the sum of the lengths of the two trains, that is, 176 yards or $\frac{1}{10}$ of a mile; the rate of gain of the faster over the slower is (40 – 35) miles, or 5 miles per hour; the question is therefore equivalent to the following: *How long will it take a particle moving at the rate of* 5 *miles an hour to move through a space of* $\frac{1}{10}$ *of a mile?*

$$t=\frac{s}{v}=\frac{\frac{1}{10}\text{ mile}}{5\text{ miles}}=\frac{1}{50}=\frac{1}{50}\text{ hour}$$

$$=\frac{60}{50}\text{ minutes}=1\tfrac{1}{5}\text{ minutes};$$

and, generally, when two such bodies are moving in the same direction

$$\text{Time taken by the faster to pass the slower}$$
$$=\frac{\text{sum of length of the moving bodies}}{\text{difference of their velocities}}.$$

Ex. 9. *A train passed a point at the entrance of a tunnel in* 11¼ *seconds, and passed completely through the tunnel which is* 682 *yards long, in* 50 *seconds: find the length of the train and its rate of speed per hour.*

Let $x=$ the length of the train in yards,

then $v=\frac{s}{t}=\frac{x}{11\frac{1}{4}}$ yards per second;

$$\text{also} \quad v=\frac{682+x}{50} \text{ yards per second ;}$$

$$\therefore \quad \frac{x}{11\frac{1}{4}}=\frac{682+x}{50},$$

from which the value of x is easily obtained, and the rate of speed can then be obtained by the method already explained.

Ex. 10. *A train travelling at the rate of* 40 *miles an hour overtakes another train* 86 *yards long, and passes it in* $1\frac{1}{5}$ *minutes; if the trains had been travelling at the same rates in opposite directions, they would have passed each other in* $4\frac{4}{5}$ *seconds: find the rate in miles per hour of the second train, and the length of the first train.*

Rate of 1st train $=\frac{40\times 1760}{60\times 60}=\frac{176}{9}$ yds. per second.

Let $x=$rate of 2nd train in yards per second, and $y=$length of 1st train in yards; then, when the trains are moving in the same direction, the faster passes the slower at the rate of $(\frac{176}{9}-x)$ yards per second; and, when they are moving in opposite directions, they pass each other at the rate of $(\frac{176}{9}+x)$ yards per second.

Sum of lengths of train $=(86+y)$ yards.

Now, when the trains are moving in the same direction, the change in their relative positions is made in 72 seconds; and, when they are moving in opposite directions, it is made in $4\frac{4}{5}$ seconds;

$$\therefore \quad 4\tfrac{4}{5}\left(\frac{176+9x}{9}\right) \text{ yards}=(86+y) \text{ yards,}$$

$$\text{and} \quad 72\left(\frac{176-9x}{9}\right) \text{ yards}=(86+y) \text{ yards ;}$$

$$\therefore \quad 4\tfrac{4}{5}\left(\frac{176+9x}{9}\right)=72\left(\frac{176-9x}{9}\right),$$

$$\text{from which} \quad x=\frac{154}{9} \text{ yards per second,}$$

that is, 35 miles per hour.

Substituting the value of x for x in the equation

$$72\left(\frac{176-9x}{9}\right)=(86+y),$$

$$\begin{aligned} \text{we have} \quad 86+y &=72\left(\frac{176}{9}-\frac{154}{9}\right) \\ &=176; \\ \therefore \quad y &=176-86 \\ &=90 \text{ yards.} \end{aligned}$$

Boat Problems.

In the case of boat problems it is considered that, if a man can row a miles per hour in still water, he can row $(a+b)$ miles per hour with a stream that flows at the rate of b miles per hour, and $(a-b)$ miles per hour against the stream (a here being assumed to be greater than b); for, while moving with the stream, he rows a miles per hour, and the stream carries him forwards b miles per hour; and, while moving against the stream, he rows a miles per hour, and the stream carries him back b miles per hour.

Obviously, his rate with the stream + his rate against it

$$=(a+b) \text{ miles} + (a-b) \text{ miles} = 2a \text{ miles per hour};$$

that is, *twice his rate in still water*.

Hence, denoting his rate with the stream by x and his rate against it by y, we have

$$\text{his rate in still water} = \tfrac{1}{2}(x+y) \text{ miles per hour.}$$

Again, his rate with the stream − his rate against it

$$=(a+b) \text{ miles} - (a-b) \text{ miles} = 2b \text{ miles},$$

that is, *twice the rate of the stream*;

$$\text{that is, rate of stream} = \tfrac{1}{2}(x-y) \text{ miles per hour.}$$

EXAMPLES. LXXXVII.

1. A particle moves n yards in t seconds: how many miles will it go in x hours at the same rate?

2. A person walks k yards in t minutes: how many hours will it take him to walk x miles at the same rate?

3. A train goes m miles in n hours: how many yards will it go in t seconds at the same rate?

4. Two villages, A and B, are 42 miles apart; at 8 a.m. one man starts from A, and walks towards B at a uniform rate of $3\frac{1}{2}$ miles an hour, and at the same time another starts from B, and travels towards A at a uniform rate of $2\frac{1}{2}$ miles an hour: when and where do they meet?

5. Two cyclists, A and B, rode from the same place to the same place, A at 15 miles an hour, and B at 12 miles an hour; B started $1\frac{1}{2}$ hours before A, and was overtaken by A exactly at the end of the journey: how long was each on the road, and how far did each travel?

6. A greyhound sees a hare at a distance of 50 yards ahead of him; both begin to run at the same instant, the dog after the hare, and the hare in the same straight line from the dog, the hare at the rate of 7 yards a second, and the dog at the rate of 10 yards a second: how far will each have run when the hare is overtaken by the dog?

7. A man rows 2 miles up a stream in 32 minutes, and back to his starting point in 24 minutes: find the rate of the stream, and the rate at which the man could row in still water.

8. Two watermen, rowing with a tide flowing at the rate of $2\frac{1}{4}$ miles an hour, row 20 miles in $3\frac{1}{5}$ hours: in what time could they row this distance against the same tide?

9. A person can row 9 miles an hour on still water, and he finds that it takes him exactly three times as long to row up as to row down a river: find the rate of the stream.

10. Two trains start at the same time from Sheffield and Leicester, and proceed towards each other at the rates of 30 and 50 miles per hour respectively; when they meet, it is found that one train has run $14\frac{1}{2}$ miles farther than the other: find the distance between Sheffield and Leicester.

11. A train leaves London for Brighton at 9 a.m., travelling at a uniform rate of 15 miles an hour; an express train leaves Brighton for London at 10 a.m., and travels at the rate of 40 miles an hour: at what time will they pass each other, and at what distance from London, the distance from London to Brighton being 50 miles?

12. A train passed a man standing at the entrance of a tunnel in $11\frac{1}{4}$ seconds, and passed completely through the tunnel in 50 seconds: if the tunnel is 682 yards long, find the length of the train and its velocity per hour.

13. If two trains running on parallel lines of a railway be respectively 150 and 200 feet long, and travel uniformly, the former at 25 and the latter at 20 miles an hour, in what time will they clear each other exactly when travelling (1) in the same direction and (2) in opposite directions?

14. A person standing on a railway platform 264 yards long noticed that a train passed the platform in 20 seconds, and himself in 8 seconds: find the length of the train, and its rate per hour.

15. A train 88 yards long overtook a person walking along the line at the rate of 4 miles an hour, and passed him completely in 10 seconds; it afterwards overtook another person, and passed him completely in 9 seconds: at what rate per hour was the second person walking?

16. A train travelling at the rate of 45 miles an hour overtakes another train 60 yards long, and passes it in 15 seconds; if the trains had been travelling in opposite directions, they would have passed each other in 3 seconds: find in miles per hour the rate of the second train, and the length of the first train.

Motion in a Circle.

Clock sums furnish typical instances of motion in a circle. An important point to notice is the fact that the space moved through by the minute hand is equal to 12 times the space moved through by the hour hand; hence, through whatever space the minute hand moves, $\frac{11}{12}$ of it is gain on the hour hand.

The minute hand makes 12 rounds while the hour hand makes 1 round;

$\therefore$ the minute hand gains 11 rounds in 12 rounds;
$\therefore$ it gains 1 round in $\frac{12}{11}$ rounds,
2 rounds in $(\frac{12}{11} \times 2)$ rounds,
n rounds in $(\frac{12}{11} \times n)$ rounds.

When n is 11, the number of rounds gained and the number traversed by the minute hand are both for the first time integral; that is, the hands are together where they started.

It is plain that there are 10 other positions, *i.e.* 11 in all, where the hands are together. If they start at 12 o'clock they are together past the 12 point $\frac{1}{11}$, $\frac{2}{11}$, $\frac{3}{11} \cdots \frac{11}{11}$ of a round, positions corresponding to the times $\frac{60}{11}$ or $5\frac{5}{11}$ minutes past 1, $10\frac{10}{11}$ minutes past 2, $16\frac{4}{11}$ minutes past 3; and so on.

It is also plain that the number of positions in which the hands come together is determined by the difference between the terms of the ratio of their speeds, that is by 11.

If the clock be a 24-hour clock, the ratio of the speeds of the hands is 24 : 1.

Here the minute hand gains 23 rounds in 24 rounds:

$\therefore$ it gains 1 round in $\frac{24}{23}$ rounds,
2 rounds in $(\frac{24}{23} \times 2)$ rounds,
n rounds in $(\frac{24}{23} \times n)$ rounds.

When n is 23, the number of rounds gained and the number traversed by the minute hand are both for the first time integral; that is, the hands are together where they started; and there are 23 different positions in which they are together. If they start at 12 o'clock, they are together past the 12 point

$\frac{1}{23}$, $\frac{2}{23}$, $\frac{3}{23}$, $\frac{4}{23} \cdots \frac{23}{23}$ of a round,

positions corresponding to the times $\frac{60}{23}$ or $2\frac{14}{23}$ minutes past 1, $(2\frac{14}{23} \times 2)$ minutes past 2, $(2\frac{14}{23} \times 3)$ minutes past 3; and so on.

The ratio of the speeds of two runners starting from the same point and running round and round a circular course is as 11 : 8; *at how many points do they come together?*

The faster gains 3 rounds in 11 rounds;

$\therefore$ he gains 1 round in $\frac{11}{3}$ or $3\frac{2}{3}$ rounds,
2 rounds in $(\frac{11}{3} \times 2)$ or $7\frac{1}{3}$ rounds,
n rounds in $(\frac{11}{3} \times n)$ rounds.

When n is 3 (that is, $11-8$), the number of rounds gained and the number traversed by the faster are both for the first time integral, and the runners are together at the starting point.

The runners are first together when the faster has run $3\frac{2}{3}$ rounds, next when he has run $7\frac{1}{3}$ rounds, and next when he has run 11 rounds. Hence, there are in the course three points at which they come together, viz., $\frac{2}{3}$ of a round past the starting point, $\frac{1}{3}$ of a round past the starting point, and at the starting point.

When two bodies move from the same point round and round a circular path in the same direction, the number of points at which they come together is represented by the difference between the terms of the ratio of their speeds when that ratio is expressed in its lowest integral terms.

Ex. 1. *A cyclist and a walker go round and round the same circular course in the same direction, the speed of the former being to that of the latter as* 6 : 1; *at what parts of the course are they together, and, if the cyclist goes once round the course in* 5 *minutes, at what intervals of time are they together?*

There are $(6-1)$ or 5 points where they come together.

In 1 round the cyclist gains $\frac{6-1}{6}$ or $\frac{5}{6}$ of a round;

$\therefore$ he gains 1 round in $\frac{6}{5}$ or $1\frac{1}{5}$ of a round,
,, 2 rounds in $2\frac{2}{5}$ rounds,
,, 3 ,, $3\frac{3}{5}$,,
,, 4 ,, $4\frac{4}{5}$,,
,, 5 ,, 6 ,,

and now they are together at the starting point.

The points of the course at which they are together are:

1st, $\frac{1}{5}$ of the course past the starting point,
2nd, $\frac{2}{5}$,, ,, ,,
3rd, $\frac{3}{5}$,, ,, ,,
4th, $\frac{4}{5}$,, ,, ,,
5th, at the starting point.

The cyclist gains a round in $\frac{6}{5}$ of 5 minutes, that is, in 6 minutes;

$\therefore$ they are together at intervals of 6 minutes.

Ex. 2. *Two men start in the same direction at the same time and from the same point in a circular course of* 680 *yards: if their rates are as* 5 : 7, *at what point in the course will the faster be* 170 *yards ahead of the slower?*

Whatever distance the faster goes, $\frac{7-5}{7}$ or $\frac{2}{7}$ is gain; in the present case the gain is 170 yards;

$$\therefore\ \tfrac{2}{7} \text{ of the required distance} = 170 \text{ yards};$$

$$\therefore\ \text{the required distance} = \tfrac{7}{2} \text{ of } 170 \text{ yards} = 595 \text{ yards},$$

that is, the point is 595 yards past the starting point.

Ex. 3. *A, B, and C start together from the same point and keep walking round a course* 90 *yards in circumference, at the rates of* 9, 5, *and* 3 *yards a second: find the time after starting when the three are together again.*

Their rates are as 9 : 5 : 3;

$$\therefore\ A \text{ gains on } B\ \tfrac{9-5}{9} \text{ or } \tfrac{4}{9} \text{ of a round in 1 round};$$

$$\therefore\ \text{he gains 1 round in } \tfrac{9}{4} \text{ rounds.}$$

A does a round in 10 seconds;

$\therefore$ in $\frac{9}{4}$ of 10 seconds, that is, in $\frac{45}{2}$ seconds, he gains 1 round on *B*.

A gains on *C* $\frac{9-3}{9}$ or $\frac{2}{3}$ of a round in 1 round;

$$\therefore\ \text{he gains 1 round in } \tfrac{3}{2} \text{ rounds};$$

$\therefore$ in $\frac{3}{2}$ of 10 seconds, that is, in 15 seconds, he gains 1 round on *C*;

$\therefore$ the walkers will be together for the first time after starting in the L.C.M. of $\frac{45}{2}$ seconds and 15 seconds, that is, in 45 seconds.

If two men start from the same point at the same time, and walk in opposite directions round and round a circular course at rates in the ratio of 5 : 7, the faster walks $\frac{7}{12}$ of a round while the slower walks $\frac{5}{12}$ of a round. Hence they meet

for the 1st time when the faster has done $\frac{7}{12}$ of a round,
for the 2nd time when he has done $(\frac{7}{12} \times 2)$ or $1\frac{2}{12}$ rounds,
for the 3rd time when he has done $(\frac{7}{12} \times 3)$ or $1\frac{9}{12}$ rounds,
for the *n*th time when he has done $(\frac{7}{12} \times n)$ rounds.

The least value of n that will give an integral number of rounds is 12. Putting 12 for n, we have

$$\tfrac{7}{12} \times n = \tfrac{7}{12} \times 12 = 7 \text{ rounds.}$$

Hence they are at the starting point when the faster has completed 7 rounds, and there are in the course 12 points at which the men meet.

When two bodies start from the same point at the same time, and move round and round a circular path in opposite directions, the number of points at which they meet is represented by the sum of the terms of the ratio of their rates when that ratio is expressed in its lowest integral terms.

Ex. 4. *At what times between* 4 *and* 5 *o'clock do the hands of a clock make an angle of* 5 *minute-spaces?*

The angle is made before and after the minute hand overtakes the hour hand. At 4 o'clock the hour hand is 20 minute-spaces ahead of the minute hand; hence, to make the required angle before overtaking the hour hand, the minute hand must gain 15 minute-spaces, and to make the angle after overtaking it, it must gain $(20+5)$ or 25 minute-spaces.

Now, to find the time when the hands of a clock are in any given position with respect to each other, it is sufficient to remember that *the minute hand gains on the hour hand* 55 *minute-spaces in* 60 *minutes, that is,* 11 *minute-spaces in* 12 *minutes.*

Hence, the time required for the minute hand to gain any number of minute-spaces is found by taking $\frac{12}{11}$ *of the number of minute-spaces.*

In the example the minute hand has to gain (1) 15 and (2) 25 minute-spaces;

$$\therefore \text{ first time} = \tfrac{12}{11} \text{ of } 15 \text{ minutes past } 4$$
$$= 16\tfrac{4}{11} \text{ minutes past } 4,$$
$$\text{and second time} = \tfrac{12}{11} \text{ of } 25 \text{ minutes past } 4$$
$$= 27\tfrac{3}{11} \text{ minutes past } 4.$$

Ex. 5. *Find* (1) *the exact time between* 5 *and* 6 *o'clock when the minute hand is* $\frac{1}{6}$ *of the circumference of the dial in advance of the hour hand;* (2) *the position of the hour hand at this time;* (3) *the angles between the hands.*

At 5 o'clock the hour hand is 25 minute-spaces in advance of the minute hand; hence, since by the question the minute hand is to reach a point $\frac{1}{6}$ of the circumference of the dial in advance of the hour hand, the former must gain $(25+\frac{1}{6}$ of $60)$ minute-spaces on the latter. We therefore have

(1) $$\text{Time} = \tfrac{12}{11} \text{ of } 35 \text{ minutes past } 5$$
$$= 38\tfrac{2}{11} \text{ minutes past } 5.$$

(2) For all times, the space moved through by the hour hand is equal to $\frac{1}{12}$ of the space moved through by the minute hand;

$$\therefore \text{ space moved through by the hour hand} = \tfrac{1}{12} \text{ of } 38\tfrac{2}{11} \text{ minute-spaces}$$
$$= 3\tfrac{2}{11} \text{ minute-spaces};$$

that is, the hour hand is $3\frac{2}{11}$ minute-spaces past the point marked 5 on dial, *i.e.* $28\frac{2}{11}$ minute-spaces past the point marked 12.

(3) At this time the minute hand is $38\frac{2}{11}$ minute-spaces past the point marked 12 on the dial;

$$\therefore \text{ no. of minute-spaces between the hands} = 38\tfrac{2}{11} - 28\tfrac{2}{11} = 10.$$

Each minute-space on the circumference is $\frac{1}{60}$ of the circumference. Now, the angle subtended at the centre of a circle by an arc that is $\frac{1}{60}$ of the circumference is an angle of $\frac{1}{60}$ of 360°, *i.e.* 6°; $\therefore$ the smaller angle between the hands is 60°, and the larger 300°.

Ex 6. *A watch gains* 1 *minute* 15 *seconds a day: if it is set right at noon on the* 12*th of November, what will be the true time when it points to noon on the following Christmas Day?*

As measured by the watch,

$$\text{time between the given dates} = (18\tfrac{1}{2} + 24\tfrac{1}{2}) = 43 \text{ days};$$

$$\text{gain of watch per day} = 75 \text{ seconds} = \tfrac{1}{1152} \text{ of a day};$$

$$\therefore \text{ time for any period as measured by this watch}$$
$$= \text{true time} + \tfrac{1}{1152} \text{ of true time};$$

$$\therefore \text{ false time of watch} = 1\tfrac{1}{1152} = \tfrac{1153}{1152} \text{ of true time};$$

$$\therefore \tfrac{1152}{1153} \text{ of false time} = \text{true time};$$

$$\therefore \text{ difference between false and true time} = \tfrac{1}{1153} \text{ of false time};$$

$$\therefore \text{ the watch is } \tfrac{1}{1153} \text{ of 43 days fast,}$$

$$\text{that is, 53 min. } 42\tfrac{234}{1153} \text{ seconds};$$

$$\therefore \text{ true time is 6 min. } 17\tfrac{919}{1153} \text{ seconds past 11 a.m.}$$

EXAMPLES. LXXXVIII.

1. Starting at the same instant, a cyclist and a runner go round and round the same circular course in the same direction, the speed of the former being to that of the latter as 7 : 2 : find at what parts of the course they are together.

2. Starting together from the winning post, two cyclists run round and round a circular course of 480 yards at speeds that are in the ratio of 11 to 8; it is noticed that the faster passes the slower every 4 minutes, and that, at the moment when the race is ended, they are for the first time together at the winning post: find (1) the time the race lasts, and (2) how many minutes' start, in a 9-mile race, the faster can afford to give the slower without losing.

3. Three riders pass a certain point on a circular track at the same instant, with velocities of 10, 12, and 15 miles an hour respectively: how long from that instant must they continue to ride round the track with these velocities before they next pass the same point abreast (the distance round the track being for each rider 704 yards)?

4. *A* and *B* start from the same point at the same time to run in opposite directions round a circular track of 425 yards; their speeds are 5 : 12: find (*a*) the number of points in the course at which they meet, (*b*) the position of each point, (*c*) how much the distance travelled by the faster exceeds that travelled by the slower when they meet for the 9th time.

5. *A* and *B* start from the same point at the same time to walk round a circular track in opposite directions; *A* goes round the course in 10 minutes, and *B* in 12 minutes: at how many points do they meet, and when are they for the first time at the starting point?

6. At what times between 4 and 5 o'clock are there 17 minute-spaces between the hands of a clock?

7. Find the times between 3 and 4 o'clock when the angle between the hands of a watch is $\frac{1}{3}$ of a right angle.

8. Two clocks are together at 12 o'clock; one of them loses 7 seconds and the other gains 8 seconds in 12 hours: when will one be half an hour before the other, and what o'clock will it then show?

9. At what times will the hands of a clock be at right angles to each other (*a*) between 6 and 7 o'clock and (*b*) between 10 and 11 o'clock?

10. A watch shows true time at 6 o'clock on Sunday morning, and at true noon on Tuesday it has gained 24 minutes: find what the true time will be when the watch shows 1 o'clock on Friday afternoon.

CHAPTER XXX.

SQUARE ROOT BY ARITHMETICAL METHODS.

Note carefully the following results:

$$(a) \qquad 1^2 = 1, \qquad 9^2 = 81;$$
$$(10)^2 = 100, \qquad (99)^2 = 9801;$$
$$(100)^2 = 10000, \ (999)^2 = 998001.$$

That is, *if a number has one or two figures, its square root has only one figure; if it has three or four figures, its square root has two figures; if it has five or six figures, its square root has three figures; and so on.*

$$(b) \qquad \cdot 1^2 = (\tfrac{1}{10})^2 = \tfrac{1}{100} = \cdot 01;$$
$$(\cdot 01)^2 = (\tfrac{1}{100})^2 = \tfrac{1}{10000} = \cdot 0001;$$
$$(\cdot 001)^2 = (\tfrac{1}{1000})^2 = \tfrac{1}{1000000} = \cdot 000001.$$

That is, *the square of a decimal contains twice as many decimal places as the decimal itself.*

The results obtained in (*a*) and (*b*) enable us to determine at a glance how many figures are contained in the square root of any number; for they show that every period (or part of a period) of two figures in the number is represented by one figure in the square root. In the number 3,96,01, for example, there are three periods, therefore there are three figures in the square root of it. In the number 5·32,22,49 there are three periods in the decimal and one in the integral part, therefore there are four figures in the square root of it, three in the decimal and one in the integral part.

The student should observe that in decimal expressions the periods are marked off, left and right, from the decimal point, not from the end of the decimal.

Ex. 1. *Extract the square root of* 1225.

In 1225 there are two periods of two figures; hence its square root has two figures, and therefore lies between 9 and 100. The tens' digit is involved in the period 12, that is, 1200.

Let a denote the tens' digit, and b the units' digit; then

$$\begin{aligned}1225 &= (a+b)^2\\ &= a^2+2ab+b^2\\ &= a^2+(2a+b)b.\end{aligned}$$

The largest square number not greater than 1200 is 900, the square root of which is 30; hence $a=30$;

$$\therefore\ 1225=(30)^2+(2a+b)b;$$

$$\therefore\ 1225-900=(2a+b)b,$$

$$\text{that is,}\quad 325=(2a+b)b;$$

$$\therefore\ b=\frac{325}{2a+b}.$$

We have now to find the number of times 325 contains $2a+b$. Compared with a, b is very small, and, for our present purpose, may be disregarded.

$$\frac{325}{2a}=\frac{325}{60}=5,\ \text{approximately.}$$

Putting 5 for b in the quotient, we have

$$b=\frac{325}{60+5}=\frac{325}{65}=5;$$

$$\text{and, since}\quad a=30,$$

$$\therefore\ a+b=35.$$

In the foregoing work, $2a$ is called the *trial divisor*, that is, the divisor by means of which we are enabled to discover the next figure of the root. *For every period after the first, the trial divisor is twice the part of the root already found.*

Ex. 2. *Extract the square root of* 978121.

Divisors.	Root.
9	97,81,21 (989
	81
188	1681
	1504
1969	17721
	17721

The first *complete divisor* is the first figure of the root. When the second figure of the root is placed in the quotient, the 9 will become 90, and 90 is then the first part of the root "already found." Twice 90, that is, 180, is therefore the trial divisor for the second figure of the root. Having obtained the next figure 8, we add it to 180, thus getting 188 as our second complete divisor. When the next figure is placed in the root, 98 will become 980, *the first part of the root already found.* Twice 980 is 1960, which is the next trial divisor. Having by trial found the next figure (9) of the root, we add it to 1960, thus getting 1969 as the third complete divisor.

The student will readily see that each succeeding *trial divisor* can be found from the preceding *complete divisor* by doubling the right-hand digit of the complete divisor. The next complete divisor is then the trial divisor with the trial root digit on its right. He will note that the figures are taken down in pairs, each pair being one of the periods of the number the square root of which is to be found.

Ex. 3. *What number multiplied by itself will produce* 391538·0329?

Obviously the number is the square root of 391538·0329. There are five periods of two digits in the number; hence there are five digits in its square root.

Long Method.

6	39,15,38,·03,29 (625·73
	36
122	315
	244
1245	7138
	6225
12507	91303
	87549
125143	375429
	375429

Short Method.

6	39,15,38,·03,29 (625·73
122	315
1245	7138
12507	91303
125143	375429

When more than half the number of digits in the root have been found by the ordinary method, the remaining digits can be found by contracted division.

6	39,15,38,·03,29 (625·73
	36
122	315
	244
1245	7138
	6225
12,5,0	913 (73.
	875
	38
	38

At this point we double the part of the root already found (625), and make the result (1250) our divisor.

It has been shown that the square of any decimal contains twice as many decimal places as the decimal itself; hence it follows that, before proceeding to extract the square root of decimals, we must make the decimal places two or some multiple of two in number by placing ciphers to the right.

If, for example, we were required to find the square root of ·016, we should write ·0160 to find the root to two decimal places, ·016000 to find it to three decimal places; and so on.

The square root of a vulgar fraction.

$$\left(\frac{5}{7}\right)^2=\frac{5\times5}{7\times7}=\frac{25}{49},$$

hence a fraction is squared by squaring its numerator and denominator; and hence, conversely, the square root of a fraction is found by extracting the square root of its numerator and denominator, thus:

$$\sqrt{\frac{25}{49}}=\frac{\sqrt{25}}{\sqrt{49}}=\frac{5}{7},$$

$$\sqrt{7\tfrac{1}{9}}=\sqrt{\frac{64}{9}}=\frac{\sqrt{64}}{\sqrt{9}}=\frac{8}{3}=2\tfrac{2}{3}.$$

The student must take care to see that every fraction is reduced to its lowest terms before he attempts to extract its square (or other) root; for, if each term contains factors that are not perfect squares, the exact square root cannot be determined. For example, each of the terms of the fraction $\frac{50}{98}$ has a factor that is a perfect square, and they both contain 2, which is not a perfect square; hence the fraction $\frac{50}{98}$ has not an exact square root, but, if the common factor 2 is cancelled out, the resulting fraction $\frac{25}{49}$ has the exact square root $\frac{5}{7}$.

Here, it will be observed, both numerators and denominators are **perfect squares**.

Ex. 4. *Extract the square root of* $\frac{5}{7}$.

Here neither the numerator nor the denominator has an exact square root; that is, 5 and 7 are not perfect squares. But

$$\frac{5}{7}=\frac{5\times7}{7\times7};\quad\therefore\ \sqrt{\frac{5}{7}}=\sqrt{\frac{5\times7}{7\times7}}=\frac{\sqrt{35}}{\sqrt{49}}=\frac{\sqrt{35}}{7}=\frac{5{\cdot}916}{7}={\cdot}845\ldots,$$

$$\text{also}\quad\sqrt{\frac{5}{7}}=\sqrt{{\cdot}714285\ldots}={\cdot}845\ldots.$$

Hence to extract the square root of a fraction, the terms of which are not perfect squares, we may either multiply the terms by a number that will make the denominator a perfect square and then divide the square root of this new denominator into the square root of the new numerator; or reduce the fraction to a decimal and extract the square root of the decimal. The first is the method that should be adopted.

Quantities the roots of which cannot be exactly obtained are called *irrational quantities* or *surds.*

$$\sqrt{2},\ \sqrt[3]{7},\ \sqrt[4]{17},\ \text{that is, } 2^{\frac{1}{2}},\ 7^{\frac{1}{3}},\ (17)^{\frac{1}{4}},$$

are examples of such numbers.

A fraction may have irrational quantities for its terms.

$$\frac{1}{\sqrt{2}} \text{ and } \frac{\sqrt{3}-1}{\sqrt{3}+1} \text{ are examples.}$$

If required to find the square root of these expressions, we proceed as follows:

$$\frac{1}{\sqrt{2}}=\frac{1\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}} \text{ (the terms multiplied by the same number)}$$

$$=\frac{\sqrt{2}}{2}=\frac{1{\cdot}4142\ldots}{2}={\cdot}7071.$$

$$\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} \text{ (the terms multiplied by the same number).}$$

We have already seen that

$$(a-b)(a-b)=a^2-2ab+b^2,$$
$$(a+b)(a-b)=a^2-b^2;$$

and these laws hold for both rational and irrational numbers;

$$\therefore\ \frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}=\frac{3-2\sqrt{3}+1}{3-1}$$

$$=\frac{4-2\sqrt{3}}{2}=2-\sqrt{3}.$$

The rest of the work may be done either by the arithmetical or by the logarithmic method.

The operation performed above is called *rationalising the denominator.* It obviously depends on the principle that multiplying the numerator and the denominator by the same number does not alter the value of a fraction.

Irrational numbers having the same root sign are said to be of the same order, and those having different root signs are said to be of different orders. Different orders can readily be expressed as surds of the same order.

$\sqrt{2}$ and $\sqrt[3]{3}$ are of the second and the third order respectively.

$$\sqrt{2}=2^{\frac{1}{2}}=2^{\frac{3}{6}}=\sqrt[6]{2^3}=\sqrt[6]{8},$$

$$\sqrt[3]{3}=3^{\frac{1}{3}}=3^{\frac{2}{6}}=\sqrt[6]{3^2}=\sqrt[6]{9}.$$

The surds being now expressed in terms of the same order, it is easy to see which is the greater, and, by means of logarithms, to find how much the one is greater than the other.

Ex. 5. *Find the value of* $\sqrt{7+3\sqrt{5}}$.

$$\begin{aligned}\sqrt{7+3\sqrt{5}}&=\sqrt{7+\sqrt{5\times 9}}=\sqrt{7+\sqrt{45}}\\&=\sqrt{7+6{\cdot}708\ldots}=\sqrt{13{\cdot}708\ldots}\\&=3{\cdot}702\ldots.\end{aligned}$$

The following results are true for all values of a and b:

(1) $\sqrt{a}\times\sqrt{b}=\sqrt{a\times b}$. (2) $\sqrt{a}\times\sqrt{a}=\sqrt{a^2}=a$.

(3) $a\sqrt{b}=\sqrt{b\times a^2}$. (4) $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$.

(5) $(\sqrt{a}+\sqrt{b})^2=a+2\sqrt{ab}+b$.

(6) $(\sqrt{a}-\sqrt{b})^2=a-2\sqrt{ab}+b$.

From (5) it follows that $\sqrt{a+2\sqrt{ab}+b}=\sqrt{a}+\sqrt{b}$,

and from (6) that $\sqrt{a-2\sqrt{ab}+b}=\sqrt{a}-\sqrt{b}$.

Ex. 6. *Find the value of* $3\sqrt{20}+4\sqrt{5}-\sqrt{\tfrac{1}{5}}$.

$$3\sqrt{20}=3\sqrt{4\times 5}=6\sqrt{5};$$

$$\sqrt{\frac{1}{5}}=\frac{\sqrt{1}}{\sqrt{5}}=\frac{\sqrt{5}}{5}=\frac{1}{5}\sqrt{5};$$

$$\begin{aligned}\therefore\ 3\sqrt{20}+4\sqrt{5}-\sqrt{\tfrac{1}{5}}&=6\sqrt{5}+4\sqrt{5}-\tfrac{1}{5}\sqrt{5}\\&=9\tfrac{4}{5}\sqrt{5}=\frac{49\sqrt{5}}{5},\end{aligned}$$

from which it is easy to find the approximate value of the given expression.

EXAMPLES. LXXXIX.

Extract the square root of:

1. 4096.
2. 59049.
3. 46656.
4. 117649.
5. 262144.
6. 390625.
7. 502681.
8. 824464.
9. 47089.
10. 1079521.
11. 49674304.
12. 11854249.

13. $5\frac{1}{16}$, $4\frac{21}{25}$, $2\frac{46}{49}$, $3\frac{13}{81}$. **14.** $109\frac{628}{729}$, $237\frac{492}{841}$, $1383\frac{21}{25}$.

15. 210·25, 144·7209, 39·175081.

16. Find to three decimal places the value of each of the following:

$$\sqrt{14{\cdot}4},\ \sqrt{{\cdot}571},\ \sqrt{{\cdot}081},\ \sqrt{\tfrac{7}{5}},\ \sqrt{\tfrac{3}{8}}.$$

Rationalise the denominators of the following:

17. $\dfrac{a}{2\sqrt{b}},\ \dfrac{3x}{4\sqrt{2y}},\ \dfrac{a}{\sqrt{2}-1},\ \dfrac{2}{\sqrt{3}+\sqrt{2}}.$

18. $\dfrac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}},\ \dfrac{3+\sqrt{5}}{3-\sqrt{5}},\ \dfrac{1}{2\sqrt{a}+3\sqrt{x}}.$

19. Find to three decimal places the square root of each of the following:

$$\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}},\ \frac{7+\sqrt{5}}{6-\sqrt{5}}-\frac{7-\sqrt{5}}{6+\sqrt{5}}.$$

20. A square field contains 3 ac. 2 ro. 7 po. 9¼ yds.: find the length of its side.

21. If the area of a field is 824464 square yards, find the length of the fence that encloses it.

22. Find the length of the fence that encloses a square field of 3 ac. 1 ro. 13 po. 5¾ yds.

23. How many poles of fencing are required to enclose a square park containing 27 ac. 12 po. 1 yd.?

24. How many poles of fencing are required to enclose a square park containing 832 ac. 2 ro. 25 po.?

CHAPTER XXXI.

QUADRATIC EQUATIONS.

AN equation that involves the square of the unknown quantity is called a *quadratic equation.*

A quadratic equation in which the unknown quantity occurs only as a square is called a *pure* quadratic.

Example : $x^2=441.$

A quadratic equation in which the unknown quantity occurs both as a square and as a first power is called an *adfected* quadratic.

Example : $x^2-2x=63.$

It must be carefully noted that the second power of both positive and negative quantities is positive.

Thus, $x\times x=x^2$, and $-x\times -x=x^2$.

Hence the root of a quadratic equation may be either positive or negative.

The equation $x^2=441$ is solved when we take the square root of each side, thus:

$$\sqrt{x^2}=\sqrt{441},$$

that is, $x=\pm 21.$

But, since the root may be either positive or negative, we write ± 21.

Ex. 1. *Find the value of x in the equation $3x^2-12=135$.*

$$3x^2-12=135\,;$$
$$\therefore\ 3x^2=147\,;$$
$$\therefore\ x^2=\tfrac{147}{3}=49\,;$$
$$\therefore\ x=\sqrt{49}=\pm 7.$$

NOTE. *The coefficient of the square of the unknown quantity must always be made unity before we proceed to extract the square root.*

Hence we have divided the terms of the equation by 3, the coefficient of x^2.

Ex. 2. *Solve the equation* $81 - 5x^2 = 0$.

$$\begin{aligned} & 81 - 5x^2 = 0\,; \\ \therefore\quad & -5x^2 = -81\,; \\ \therefore\quad & 5x^2 = 81\,; \\ \therefore\quad & x^2 = \tfrac{81}{5}\,; \\ \therefore\quad & x = \frac{\sqrt{81}}{\sqrt{5}} = \pm\frac{9}{\sqrt{5}}, \end{aligned}$$

the value of which can be found approximately by methods already explained.

NOTE. There is no square root of a negative quantity; hence we have changed $-5x^2 = -81$ into $5x^2 = 81$ by mentally dividing the terms by -1.

EXAMPLES. XC.

Solve the following equations:

1. $4x^2 - 84 = 400$.

2. $5x^2 - 76 = 769$.

3. $49 - 3x^2 = 0$.

4. $676 - 7x^2 = 0$.

5. $3x^2 - 5 = \dfrac{8x^2}{3} + 7$.

6. $x^2 - \dfrac{3x^2 - 2}{5} = 3 - \dfrac{2x^2 - 5}{3}$.

7. $\dfrac{2x^2 + 10}{15} = 7 - \dfrac{50 + x^2}{25}$.

8. $\dfrac{x^2}{5} - \dfrac{x^2}{15} + \dfrac{x^2}{25} = 4\tfrac{1}{3}$.

The Solution of Adfected Quadratic Equations.

$$(a+b)^2 = a^2 + 2ab + b^2,$$
$$(a-b)^2 = a^2 - 2ab + b^2.$$

Hence $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$ are perfect squares.

Note that $2b$ is the coefficient of a in the second term and that

$$(\tfrac{1}{2} \text{ of } 2b)^2 = b^2.$$

Hence each of the expressions $a^2 + 2ab$ and $a^2 - 2ab$ is made a perfect square by the *addition to it of the square of half the coefficient of the quantity a in the second term.*

Again,
$$(x+5)^2 = x^2 + 10x + 25,$$
$$(x-7)^2 = x^2 - 14x + 49,$$

where 25 is seen to be the square of half the coefficient of x in the second term of the first identity, and 49 to be the square of half the coefficient of x in the second term of the second identity.

And always, if we have an expression of the form x^2+10x or x^2-14x, we can make it a perfect square by adding to it *the square of half the coefficient of the second term.*

Ex. 1. *Solve the equation* $x^2+4x=117$.

Adding to each side of the equation the square of half the coefficient of x, we have

$$x^2+4x+2^2=117+2^2;$$

that is, $$(x+2)^2=121;$$

$$\therefore\ x+2=\sqrt{121}=\pm 11.$$

We now have two simple equations:

$$x+2=+11 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and $$x+2=-11 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From these we find the value of x thus:

from (1) $x=11-2=9$;

from (2) $x=-11-2=-13$.

Hence the roots of the equations are 9 and -13.

Ex. 2. *Solve the equation* $9x-5x^2-2\frac{1}{4}=0$.

Putting the terms without x on the right-hand side of the equation, we have

$$9x-5x^2=2\tfrac{1}{4}.$$

The term with x^2 must be made positive; hence we write the equation thus;

$$5x^2-9x=-2\tfrac{1}{4}.$$

We must make the coefficient of x^2 unity; hence we divide throughout by 5:

$$x^2-\frac{9x}{5}=-\frac{2\frac{1}{4}}{5}=-\frac{9}{20}.$$

Then $$x^2-\frac{9x}{5}+\left(\frac{9}{10}\right)^2=-\frac{9}{20}+\left(\frac{9}{10}\right)^2$$

$$=\frac{-45+81}{100}$$

$$=\frac{36}{100};$$

$$\therefore\ x-\frac{9}{10}=\frac{\sqrt{36}}{\sqrt{100}}=\pm\frac{6}{10}\ \text{(sq. root of each side taken)};$$

$$\therefore\ \text{one root of } x \text{ is } \left(\frac{6}{10}+\frac{9}{10}\right),\ \text{that is, } 1\tfrac{1}{2},$$

$$\text{and the other root is } \left(\frac{9}{10}-\frac{6}{10}\right),\ \text{that is, } \frac{3}{10}.$$

We may now bring together the steps we take to solve an adfected quadratic equation:

(1) *If necessary, transpose the terms of the equation, placing on the left-hand side the terms having in them* x^2 *and* x*, and the term or terms without* x *on the right-hand side.*

(2) *Make the coefficient of* x^2 *unity and positive by dividing throughout by the coefficient of* x^2.

(3) *Add to each side of the equation the square of half the coefficient of* x.

(4) *Extract the square root of each side.*

(5) *Solve the resulting simple equations.*

EXAMPLES. XCI.

Solve the equations:

1. $x^2=3x+10$.

2. $x^2-5x+4=0$.

3. $x^2-8x+3=12$.

4. $3x^2+2x-9=76$.

5. $7x-x^2=6$.

6. $x^2+\frac{x}{2}=3$.

7. $7x^2-11x-6=0$.

8. $7x=6-3x^2$.

9. $x^2=34+\frac{x}{3}$.

10. $5(x^2+1)=22+3(x-1)$.

Problems Leading to Quadratic Equations.

In our solutions of problems leading to quadratics, we must remember that *every quadratic equation has two roots*; hence it is necessary, when the roots are found, to ascertain by trial whether they do or do not satisfy the conditions of the problem.

Ex. 1. *The length of a rectangular plot of ground exceeds its width by* 4 *yards, and its area is* 572 *square yards: find its dimensions.*

Let x denote its width in yards, then its length is $(x+4)$ yards, and its area $x\,(x+4)$ square yards;

$$\therefore\ x^2+4x=572;$$

$$\therefore\ x^2+4x+(2)^2=572+(2)^2=576;$$

$$\therefore\ x+2=\sqrt{576}=\pm 24;$$

$$\left.\begin{array}{rl} \therefore & x=24-2=22, \\ \text{or} & x=-24-2=-26. \end{array}\right\}$$

-26 does not satisfy the conditions of the problem;

$$\therefore\ \text{width}=22 \text{ yards, and length}=26 \text{ yards.}$$

Ex. 2. *A cyclist travelled to a place* 80 *miles from his starting point; on his return journey he increased his speed by* 4 *miles an hour, and his journey home took* 1 *hour less than his journey out: find the rates at which he travelled, and the time taken each way.*

Let x denote in miles his rate per hour to the place, then his homeward rate was $(x+4)$ miles per hour;

$$\therefore \text{ time of outward journey} = \frac{80}{x} \text{ hours,}$$

$$\text{and time of home journey} = \frac{80}{x+4} \text{ hours;}$$

$$\therefore \frac{80}{x} - \frac{80}{x+4} = 1;$$

$$\therefore 80(x+4) - 80x = x(x+4),$$

$$\text{that is, } 320 = x^2 + 4x,$$

$$\text{that is, } x^2 + 4x = 320;$$

$$\therefore x^2 + 4x + (2)^2 = 320 + (2)^2 = 324;$$

$$\therefore x + 2 = \sqrt{324} = \pm 18;$$

$$\left.\begin{array}{l}\therefore x = 18 - 2 = 16, \\ \text{or } x = -18 - 2 = -20.\end{array}\right\}$$

-20 does not satisfy the conditions of the problem;

$\therefore x = 16$, and rate home $= 16 + 4 = 20$ miles an hour,

and times $= 5$ and 4 hours respectively.

Ex. 3. *When* 24 *is added to a number, and the square root of the sum extracted, the square root is found to be less than the number by* 18; *find the number.*

Let x denote the number,

$$\text{then } \sqrt{x+24} = x - 18;$$

$$\therefore x + 24 = (x-18)^2$$

$$= x^2 - 36x + 324.$$

Transposing and simplifying, we have

$$x^2 - 37x = -300;$$

$$\therefore x^2 - 37x + \left(\tfrac{37}{2}\right)^2 = -300 + \left(\tfrac{37}{2}\right)^2$$

$$= \tfrac{169}{4};$$

$$\therefore x - \frac{37}{2} = \frac{\sqrt{169}}{\sqrt{4}} = \pm \frac{13}{2};$$

$$\left.\begin{array}{l}\therefore x = \tfrac{13}{2} + \tfrac{37}{2} = \tfrac{50}{2} = 25, \\ \text{or } x = -\tfrac{13}{2} + \tfrac{37}{2} = \tfrac{24}{2} = 12.\end{array}\right\}$$

12 does not satisfy the conditions of the problem;

$\therefore$ 25 is the required number.

EXAMPLES. XCII.

1. The sum of two numbers is 15, and their product is 56: find the numbers.

2. Find a number that is less than its square by 110.

3. The difference of two numbers is 16, and their product is 377: find the numbers.

4. The length of a rectangle is 6 yards more than its breadth, and its area is 247 square yards: find its dimensions.

5. A man cycled 200 miles at a uniform rate; if his speed had been 5 miles an hour greater, he would have finished his journey two hours sooner: find the rate at which he travelled.

6. I bought a number of sheep for £6. 8*s.*, which I afterwards sold at 18 shillings per head, and thereby gained as much as one sheep cost me: how many did I buy?

7. The fence that encloses a rectangular garden measures 148 yards, and the area of the garden is 1200 square yards: find the length and breadth of the garden.

8. The sum of the length and breadth of a field is 76 chains, and the area of the field is 142·8 acres: find its length and breadth.

9. A path of uniform width surrounds a rectangular lawn 25 yards by 20 yards, and the area of the path is 424 square yards: find its width.

10. A person bought a number of cows for £80; if he had bought 4 more for the same money, he would have paid £1 less for each cow: how many did he buy?

CHAPTER XXXII.

CUBE ROOT BY ARITHMETICAL METHODS.

The following results should be carefully noted:

$$(a)\quad 1^3 = 1,\ 9^3 = 729;$$
$$10^3 = 1{,}000,\ 99^3 = 970{,}299;$$
$$100^3 = 1{,}000{,}000,\ 999^3 = 997{,}002{,}999.$$

That is, *a number of one digit has in its cube not fewer than one, nor more than three digits; a number of two digits has in its cube not fewer than four, nor more than six digits; a number of three digits has in its cube not fewer than seven, nor more than nine digits.*

$$(b)\quad \cdot 1^3 = (\tfrac{1}{10})^3 = \tfrac{1}{1000} = \cdot 001;$$
$$\cdot 01^3 = (\tfrac{1}{100})^3 = \tfrac{1}{1000000} = \cdot 000001;$$
$$\cdot 001^3 = (\tfrac{1}{1000})^3 = \tfrac{1}{1000000000} = \cdot 000000001.$$

Hence the cube of any decimal contains three times as many decimal places as the decimal itself.

The results obtained in (*a*) and (*b*) enable us to determine at a glance the number of figures in the cube root of any integral or decimal number that has an exact cube root; for they show that every period (or part of a period) of three figures (numbered from the units in the integral and from the tenths in the decimal part) in the cube represents one figure in the cube root.

The following table of squares and cubes should be thoroughly known:

Number -	1	2	3	4	5	6	7	8	9	10	11	12
Square -	**1**	**4**	**9**	**16**	**25**	**36**	**49**	**64**	**81**	**100**	**121**	**144**
Cube -	**1**	**8**	**27**	**64**	**125**	**216**	**343**	**512**	**729**	**1000**	**1331**	**1728**

Ex. 1. *Find the cube root of* 571787.

The number contains two periods of three digits, and therefore its cube root is a number of two digits.

Let a denote the tens' digit and b the units' digit;

$$\text{then}\quad (a+b)^3=571787.$$

$$\text{Now,}\quad (a+b)^3=a^3+3a^2b+3ab^2+b^3,$$

which the student should verify by actual multiplication, and memorise.

Find the nearest perfect cube below 571000; the number is 512000, which is the cube of 80;

$$\text{hence}\quad a^3=(80)^3=512000;$$

$$\therefore\ 512000+3a^2b+3ab^2+b^3=571787;$$

$$\begin{aligned}\therefore\ 3a^2b+3ab^2+b^3&=571787-512000\\&=59787.\end{aligned}$$

To find the value of b, divide $3a^2$, that is, $3\times(80)^2$, into the remainder 59787. $3\times(80)^2=19200$, which seems to be contained 3 times in 59787. Try 3; thus:

$$\begin{aligned}3a^2b+3ab^2+b^3&=3\times(80)^2\times 3+3\times 80\times 3^2+3^3\\&=57600+2160+27\\&=59787;\end{aligned}$$

$$\text{hence}\quad b=3.$$

The working may be set out as follows:

$$\begin{array}{rr|l}
a^3=(80)^3= & 512000 & 571787\ \big(\ a+b \\
 & & \underline{512000}\ \big(\ 80+3=83 \\
3a^2b=3\times(80)^2\times 3= & 57600 & 59787=3a^2b+3ab^2+b^3 \\
3ab^2=3\times 80\times 3^2= & 2160 & \\
b^3=3^3= & \underline{27} & \\
\therefore\ 3a^2b+3ab^2+b^3= & 59787 & \underline{59787}
\end{array}$$

$$\therefore\ \sqrt[3]{571787}=83.$$

$3a^2$, that is, $3\times(80)^2$ or 19200 is called the *trial divisor*, because by dividing it into the remainder 59787 we ascertain the next digit of the root. $(80)^2\times 3$ is three times the square of the root already found; *hence three times the square of the root already found is the trial divisor.*

Ex. 2. *Extract the cube root of* 653972032.

The number contains three periods of three digits; hence its cube root consists of three digits. The nearest perfect cube below 653 is 512, which is the cube of 8; 8 is therefore the first digit of the root.

$$\begin{array}{rrl}
 & & 653,972,032\ (\ 868 \\
 & 8^3= & \underline{512} \\
 & & 141972 \\
3\times(80)^2\times 6= & 115200 & \\
3\times 80\times 6^2= & 8640 & \Big\}=\underline{124056} \\
6^3= & 216 & \\
 & & 17916032
\end{array}$$

For convenience, ciphers are, as here, usually omitted.

$$\left.\begin{array}{rr} 3\times(860)^2\times 8= & 17750400 \\ 3\times 860\times 8^2= & 165120 \\ 8^3= & 512 \end{array}\right\}=17916032\,;$$

$$\therefore\ \sqrt[3]{653972032}=868.$$

NOTES ON THE WORKING.

The trial divisor for the second digit of the root is $(80)^2\times 3$, that is, *three times the square of the part of the root already found.*

The trial divisor for the third digit of the root is $(860)^2\times 3$, that is, *three times the square of the part of the root already found.*

It is seen that *the square of the part of the root already found* is a *variable quantity* dependent on the number of digits in the part of the root already found.

Let a denote this variable, and b the next digit of the root; then $3a^2$ is the trial divisor, and $3a^2b$ the trial divisor × the next digit of the root; and the rest of the divisor is $3ab^2+b^3$. Thus, for every root digit after the first, the *complete* divisor is $3a^2b+3ab^2+b^3$.

It is obvious therefore that the theorem

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$

enables us to find, either exactly or approximately, the cube root of any number however large.

We have seen that the cube of any decimal contains three times as many decimal places as the decimal itself; hence it follows that in extracting the cube root we must first of all make the decimal places three or some multiple of three in number by placing ciphers to the right.

Ex. 3. *Let it be required to extract the cube root of ·3 to three decimal places.*

Since the root is to be extracted to three places of decimals there must be three periods of three figures in the decimal the cube root of which we are to extract; therefore we must affix eight ciphers to ·3.

$$\begin{array}{lr} & \cdot 300{,}000{,}000\ (\ \cdot 669 \\ & 6^3=\underline{216}\phantom{{,}000{,}000\ (\ \cdot 669} \\ & 84000 \\ \left.\begin{array}{rr} 3\times 60^2\times 6= & 64800 \\ 3\times 60\times 6^2= & 6480 \\ 6^3= & 216 \end{array}\right\}= & \underline{71496} \\ & 12504000 \\ \left.\begin{array}{rr} 3\times 660^2\times 9= & 11761200 \\ 3\times 660\times 9^2= & 160380 \\ 9^3= & 729 \end{array}\right\}= & \underline{11922309} \\ & 581691 \end{array}$$

The decimal point is, as here, usually omitted.

Such examples should, as a rule, be worked by means of logarithms, thus:

$$\log \cdot 3 = \bar{1}\cdot 4771\,;$$
$$\begin{aligned} \therefore\ \log \sqrt[3]{\cdot 3} &= \tfrac{1}{3} \text{ of } (\bar{1}\cdot 4771) \\ &= \tfrac{1}{3} \text{ of } (\bar{3} + 2\cdot 4771) \\ &= \bar{1}\cdot 8257\,; \end{aligned}$$
$$\text{antilog of } \bar{1}\cdot 8527 = \cdot 6694\,;$$
$$\sqrt[3]{\cdot 3} \text{ to three places} = \cdot 669.$$

Cube Root of Fractions.

Ex. 4. *Find the cube root of* $\frac{27}{125}$ *and of* $\frac{4}{5}$ *to three decimal places.*

The first thing to observe is that both fractions are expressed in their lowest terms, and the second, that the terms of the first are perfect cubes and that neither of the terms of the second is a perfect cube.

$$\sqrt[3]{\frac{27}{125}} = \frac{\sqrt[3]{27}}{\sqrt[3]{125}} = \frac{3}{5},$$

where we have found the cube root of the fraction by extracting the cube root of each of its terms.

$$\sqrt[3]{\frac{4}{5}} = \frac{\sqrt[3]{4}}{\sqrt[3]{5}} = \frac{\sqrt[3]{4} \times \sqrt[3]{5} \times \sqrt[3]{5}}{\sqrt[3]{5} \times \sqrt[3]{5} \times \sqrt[3]{5}} = \frac{\sqrt[3]{100}}{5}.$$

Or thus: $\frac{4}{5} = \cdot 8\,;\quad \therefore\ \sqrt[3]{\frac{4}{5}} = \sqrt[3]{\cdot 8}.$

The value of $\frac{\sqrt[3]{100}}{5}$ and $\sqrt[3]{\cdot 8}$ can be found either by the logarithmic or by the arithmetical method.

Roots that are Multiples of Square and Cube Roots.

$$a^4 = (a^2)^2\,; \qquad \therefore\ \sqrt[4]{a^4} = \sqrt{\sqrt{a^4}}.$$
$$a^6 = (a^3)^2 \text{ or } (a^2)^3\,; \qquad \therefore\ \sqrt[6]{a^6} = \sqrt{\sqrt[3]{a^6}} \text{ or } \sqrt[3]{\sqrt{a^6}}.$$
$$a^9 = (a^3)^3\,; \qquad \therefore\ \sqrt[9]{a^9} = \sqrt[3]{\sqrt[3]{a^9}}.$$

Arithmetical illustrations:

$$\sqrt[4]{16} = \sqrt{\sqrt{16}} = \sqrt{4} = 2.$$
$$\sqrt[6]{64} = \sqrt[3]{\sqrt{64}} = \sqrt[3]{8} = 2 \text{ or } \sqrt{\sqrt[3]{64}} = \sqrt{4} = 2.$$
$$\sqrt[9]{134217728} = \sqrt[3]{\sqrt[3]{134217728}} = \sqrt[3]{512} = 8.$$

Ex. 5. *A cubical block of granite contains* 274625 *cubic inches: find the area of one of its sides.*

Since its three dimensions are equal;

$\therefore$ $\sqrt[3]{274625}$ inches = the length of a side = 65 inches;

$\therefore$ area of a side = (65×65) sq. inches = 4225 sq. inches

= 3 sq. yds. 2 sq. ft. 49 sq. inches.

EXAMPLES. XCIII.

Find the cube root of:

1. 15625.
2. 19683.
3. 42875.
4. 175616.
5. 571787.
6. 42508549.
7. 15813251.
8. 1030301.
9. 8120601.
10. 41063625.
11. 179597069288.
12. 2924·207.
13. 2248·091.
14. ·000085184.
15. ·000079507.

16. Find, to 3 places of decimals, the cube root of $\frac{1}{10}$, $\frac{1}{5}$, $\frac{3}{10}$, ·0047, ·00037.

17. Find, to 4 places of decimals, the cube root of $7\frac{1}{5}$, $9\frac{1}{6}$, $5\frac{4}{9}$.

18. Find the sixth root of 729, 15625, and 531441.

19. Find the ninth root of 262144, and 5159780352.

20. A cubical block of stone contains 54872 cubic inches: find the length of one of its sides.

21. A cubical vessel contains 74088 cubic inches of air: find the length of one of its sides.

22. Find the area of one side of a cubical block that contains 300763 cubic feet.

23. What is the surface of a cube the cubical content of which is 91125 cubic feet?

CHAPTER XXXIII.

PROGRESSION.

Arithmetical Progression.

EXAMINE the series 3, 5, 7, 9, 11, 13. It is seen that each succeeding term is greater by 2 than the one immediately preceding it. In the series 17, 14, 11, 8, 5, 2 each succeeding term is less by 3 than the one immediately preceding it. Numbers that increase or decrease in this way by a **constant difference** are said to be in **arithmetical progression**. The first of the foregoing series is an **ascending** progression and the second a **descending** progression. In the first the constant difference is 2; in the second it is -3. Simple series like these are easily summed; but to sum a series such as $\frac{5}{6}$, $\frac{1}{2}$, $\frac{1}{6}$, ... to a large number of terms involves great labour, unless the summation is performed by some generalised process. Let us try to discover such a process.

$$\begin{array}{lrrrrrr}
\text{Write the first series thus:} & 3, & 5, & 7, & 9, & 11, & 13 \\
\text{Then reverse it thus:} & 13, & 11, & 9, & 7, & 5, & 3 \\ \cline{2-7}
\text{Then sum the series thus:} & \multicolumn{6}{r}{16+16+16+16+16+16}
\end{array}$$

$$\begin{aligned}
&= \text{the sum of the two series;} \\
\therefore\ \text{sum} \times 2 &= (16 \times 6) = (3+13)6 \\
&= (\text{1st term} + \text{last term}) \times \text{number of terms;} \\
\therefore\ \text{sum} &= \frac{(\text{1st term} + \text{last term}) \times \text{number of terms}}{2} \\
&= \frac{(3+13)6}{2} = 48;
\end{aligned}$$

and this method holds good for all cases; hence denoting

the first term by a,
the last term by l,
the number of terms by n,
the sum of all the terms by s,

we may express our result in the following general form:

$$s = \frac{n}{2}(a+l).$$

Let a be the first term of any series of numbers in arithmetical progression, and d the common difference; then

the 2nd term $= a + d$,
the 3rd term $= a + 2d$,
the 4th term $= a + 3d$, and so on;

the terms of the series differing only in the coefficient of d, and each coefficient of d being always less by 1 than the number of the term in the series. Thus the coefficient of d in the *third* term is 2, and in the *fourth* term it is 3. In the nth term it would obviously be $n - 1$. Hence the nth term of any series of numbers in arithmetical progression is

$$a + (n-1)d.$$

When n denotes the number of terms, and l the last term, we have

$$l = a + (n-1)d.$$

Putting this value of l for l in the expression $s = \frac{n}{2}(a + l)$, we have

$$s = \frac{n}{2}\{a + a + (n-1)d\}$$

$$= \frac{n}{2}\{2a + (n-1)d\}.$$

For convenience of reference, we bring together the results we have obtained:

$$l = a + (n-1)d; \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$s = \frac{n}{2}(a + l); \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$s = \frac{n}{2}\{2a + (n-1)d\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

We now proceed to apply these equations.

Ex. 1. *If a man plants* 6 *trees the first year*, 9 *the second*, 12 *the third, and so on for* 19 *years, how many will he have planted in* 19 *years?*

Here we have to find the sum of the series 6, 9, 12 ... to 19 terms.

$$l = a + (n-1)d, \text{ and } d = 3;$$

$$\therefore\ l = 6 + (19-1)3 = 60;$$

$$s = \frac{n}{2}(a + l) = \tfrac{19}{2}(6 + 60) = 627.$$

627 trees. *Ans.*

By using the equation, $s = \frac{n}{2}\{2a + (n-1)d\}$, we can obtain the value of s without finding the value of l, thus:

$$s = \tfrac{19}{2}\{12 + (19-1)3\} = 627.$$

Ex. 2. *A man agreed to pay a debt by monthly instalments in arithmetical progression; his fifth payment was* 14 *shillings, and his seventeenth* £2. 10*s.*: *find how much larger each succeeding payment was than the one immediately preceding it, and how much the man had paid at the end of the seventeenth month.*

We have to find the series the fifth term of which is 14 and the seventeenth 50.

$$\begin{aligned} \text{5th term} &= a + 4d = 14 \text{ shillings} \\ \text{17th term} &= a + 16d = 50 \text{ shillings} \\ \hline \therefore \text{ by subtraction} \quad & 12d = 36 \text{ shillings} \end{aligned}$$

$\therefore\ d = 3$ shillings = increase in each succeeding payment.

$$\text{Also} \quad a + 3s. \times 4 = 14 \text{ shillings};$$

$$\text{that is,} \quad a + 12s. = 14 \text{ shillings};$$

$$\therefore\ a = 2 \text{ shillings} = \text{1st payment.}$$

$$\text{Sum} = \frac{n}{2}(a+l) = \frac{17}{2}(2+50) \text{ shillings}$$

$$= 442 \text{ shillings} = £22.\ 2s.;$$

$\therefore$ amount paid at end of 17th month = £22. 2*s.*

Ex. 3. *A man cycles for* 12 *days, riding* 41 *miles the first day,* 38 *miles the second day,* 35 *miles the third day, and so on: find the length of his last day's journey, and the total distance travelled in the* 12 *days.*

We have to find the last term of the series 41, 38, 35 ... to 12 terms. The series being a descending one, the common difference is negative, namely, -3. In a descending series

the expression $l = a + (n-1)d$ becomes $l = a - (n-1)d$,

for $(n-1) \times -d = -(n-1)d$;

hence in our example $l = 41 - (12-1)3 = 8$;

$\therefore$ length of last day's journey = 8 miles.

$$\text{Sum} = \frac{n}{2}(a+l) = \frac{12}{2}(41+8) = 294;$$

$\therefore$ distance travelled in 12 days = 294 miles.

Ex. 4. *Find how long it will take a man to discharge a debt of* £41 *by paying* 3 *shillings the first week,* 7 *shillings the second week,* 11 *shillings the third week, and so on.*

This is equivalent to the question: "How many terms of the series 3, 7, 11 ... must be taken to amount to 820?" for £41 = 820 shillings.

$$s = \frac{n}{2}\left\{2a + (n-1)d\right\} = \frac{n}{2}\left\{6 + (n-1)4\right\}$$

$$= 2n^2 + n.$$

By the question $s = 820$ shillings;

$$\therefore\ 2n^2 + n = 820;$$

$$\therefore\ n^2+\tfrac{1}{2}n=410;$$

$$\therefore\ n^2+\tfrac{1}{2}n+(\tfrac{1}{4})^2=410+(\tfrac{1}{4})^2$$

$$=\tfrac{6561}{16};$$

$$\therefore\ n+\frac{1}{4}=\frac{\sqrt{6561}}{\sqrt{16}}=\pm\frac{81}{4};$$

$$\therefore\ n=\tfrac{81}{4}-\tfrac{1}{4}=20;$$

$$\text{or } n=-\frac{81}{4}-\frac{1}{4}=-\frac{82}{4}\ \text{[which is inadmissible]};$$

∴ the debt will be discharged in 20 weeks.

The student should verify the following results:

sum of 1, 3, 5 = $3^2=n^2$,
sum of 1, 3, 5, 7 = $4^2=n^2$,
sum of 1, 3, 5, 7 ... to 45 terms = $45^2=n^2$.

Hence **the sum of the series of consecutive odd numbers beginning with unity = the square of the number of terms.** In this series, therefore, n is easily found.

Notice further that the common difference is double of the first term. Now in all series in which this is the case the sum of the series = $n^2\times a$. If a is unity, as above, then the sum is n^2.

Let a be 3; then we have the series 3. 9, 15, 21, ...

$$3+9+15+21=48=4^2\times 3=n^2\times 3.$$

Hence, if a is 3 in a series of 4 terms, the sum of the terms $=3n^2$.

Generally, if the first term is x, and the common difference $2x$, the sum of the series is n^2x.

The series 16, 48, 80, and so on, represents the number of feet (within a trifle) through which a body, free to fall, passes in successive seconds. The common difference is 32, which is double of the first term; hence the space passed through by a falling body in n seconds = $n^2\times 16$. Suppose the body to have been falling 10 seconds,

then space passed through = $(10^2\times 16)$ feet.

When three numbers are in arithmetical progression the middle one is called the **arithmetic mean** of the two others. If we take any three consecutive terms of the series 2, 5, 8, 11, 14 ...

and divide the sum of the 1st and 3rd terms by 2, the quotient will be the middle term thus:

$$2,\ 5,\ 8;\ \tfrac{2+8}{2} = 5$$

$$5,\ 8,\ 11;\ \tfrac{5+11}{2} = 8$$

$$8,\ 11,\ 14;\ \tfrac{8+14}{2} = 11.$$

Take the terms of the series

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d, \text{ etc.}$$

The sum of the first and third terms $= a + a + 2d = 2a + 2d$; $\therefore$ half the sum of the first and third terms $= a + d$, which is the second term of the series.

The sum of the first and fifth terms $= a + a + 4d = 2a + 4d$; $\therefore$ half the sum of the first and fifth terms $= a + 2d$, which is the third term of the series, the term *midway* between the first and fifth terms

The sum of the second and sixth terms $= a + d + a + 5d$

$$= 2a + 6d;$$

$\therefore$ half the sum of the second and sixth terms $= a + 3d$, which is the fourth term of the series, the term *midway* between the second and sixth terms.

And these results are quite general; hence the middle term between any two terms in arithmetical progression is half the sum of the two terms.

By an extension of meaning all the terms lying between the first and last terms of an arithmetical progression are called arithmetic means. It is easy to insert any given number of means between two given terms.

Ex. 5. *Insert* 16 *arithmetical means between* 6 *and* 57.

2 extremes + 16 means = 18 terms, of which 6 is the first and 57 the last.

$$\text{The 18th term} = a + 17d = 57;$$

$$\therefore\ 6 + 17d = 57,$$

$$\text{that is,}\quad 17d = 51;$$

$$\therefore\ d = 3;$$

$\therefore$ the series is 6, 9, 12, 15, ... 57, and the required means are 9, 12, 15, ... 54.

EXAMPLES. XCIV.

1. Find the last term and the sum of the series 5, 8, 11 ... to 25 terms.

2. Find the last term and the sum of the series 3, $4\frac{1}{2}$, 6, $7\frac{1}{2}$ ... to 45 terms.

3. Insert four arithmetic means between 1 and 243.

4. Insert seven arithmetic means between 1 and 33.

5. Find the series of which the 2nd term is $\frac{1}{2}$ and the 55th term 5·8.

6. Find the series of which the 59th term is 70 and the 66th term 84.

7. Find the series of which the sum of the 3rd and 8th terms is 31 and the sum of the 5th and 10th terms 43.

8. The common difference is -9, the first term 193, and the number of terms 15: find the last term and the sum of the series.

9. Find the 16th term of the series 1, $1\frac{1}{2}$, 2, and so on.

10. The sum of a series of 16 terms is 142 and the last term $10\frac{3}{4}$: find the common difference.

11. If a person travels 30 miles the first day, and each succeeding day a quarter of a mile less than he did on the day before, how far will he travel in 30 days?

12. How many of the terms 1, 3, 5, 7, and so on, make 9604?

13. A man travelled 9 days, each day's journey being $2\frac{1}{2}$ miles greater than the previous day's; on the last day he travelled $32\frac{1}{2}$ miles: what was the first day's journey?

14. A man bought 13 sheep, giving for the first sheep 3 shillings, and for each succeeding sheep 6 shillings more than for the one immediately preceding it: what did he pay for the last sheep and what for all the sheep?

15. The youngest of 12 persons is 8 years old and the oldest 85; their ages being in arithmetical progression, find at what intervals they were born.

16. A person received 6 shillings for his first day's work, 6*s.* 3*d.* for his second day's, 6*s.* 6*d.* for his third day's, and so on, till he received 13*s.* 3*d.* a day: how many days did he work, and how much did he receive in all?

17. Two clerks are engaged at the same time; *A* chooses a salary of £100 a year increasing £10 annually; *B* chooses a salary of £50 per half year increasing £5 half-yearly: which makes the better bargain, and how much more than the other will he have received at the end of 20 years?

18. A man spends on the first day 19 shillings, twice that amount the next day, and 19 shillings additional every day till he exhausts his fortune by spending on the last day £190: find his fortune and the number of days he took to spend it.

19. A falling body descends 16·1 ft. in the first second of its fall, 48·3 ft. in the second, 80·5 ft. in the third, and so on: determine the space through which it would fall in the 19th second of its fall.

20. How many times does a clock strike in 6 days?

Geometrical Progression.

Quantities are said to be in geometrical progression when they increase or decrease by a *constant factor*. The following are examples:

(1) $2, 4, 8, 16, \dots;$

(2) $1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \cdots;$

(3) $a, ar, ar^2, ar^3, \dots.$

In (1) the constant factor is 2, in (2) it is $\frac{1}{4}$, and in (3) it is r.

To find the constant factor we divide any term by the term that precedes it. In every series in geometrical progression the quotient obtained by the division of any term by the term that precedes it is always the same. If the terms of a series do not satisfy this test, they are not in geometrical progression.

The *constant factor* is called the *common ratio* of the series, and is generally denoted by r.

Let a be the first term of a geometrical progression, and r the common ratio; then

the 2nd term $= ar$,

the 3rd term $= ar^2$,

the 4th term $= ar^3$, and so on;

the terms of the series differing only in the index of r, and the index of r being always 1 less than the number of the term in the series. Thus, the index of r in the *third* term is 2, and in the *fourth* term it is 3. In the nth term it would be $n-1$. Hence the nth term of any series of numbers in geometrical progression is

$$ar^{n-1}.$$

When n denotes the number of terms, and l the last term, we have

$$l = ar^{n-1}.$$

From this equation we have

$$(1) \quad a = \frac{l}{r^{n-1}};$$

$$(2) \quad r^{n-1} = \frac{l}{a};$$

$$(3) \quad r = \sqrt[n-1]{\frac{l}{a}}.$$

Ex. 1. *Find the seventh term of the series* 3, 9, 27, ...

Here $n=7$, and $r=3$;

$$\therefore\ l=ar^{n-1}=3\times 3^6=2187.$$

Ex. 2. *The 8th term of a geometrical series is* 4374, *and the 4th term* 54: *find the series.*

$$\text{The 8th term}=ar^7=4374,$$

$$\text{the 4th term}=ar^3=54;$$

$$\therefore\ \frac{ar^7}{ar^3}=\frac{4374}{54};$$

$$\therefore\ r^4=\frac{4374}{54}=81;$$

$$\therefore\ r=\sqrt[4]{81}=\sqrt{\sqrt{81}}=3,$$

$$\text{whence}\quad a\times 3^3=54;$$

$$\therefore\ a=\frac{54}{3^3}=2;$$

$$\therefore\ \text{the series is } 2,\ 6,\ 18,\ \ldots.$$

When three quantities are in geometrical progression, the middle quantity is said to be the *geometric mean* between the two others. Thus, if a, b, and c be in geometrical progression, b is the *geometric mean* between a and c.

We have seen that, when quantities are in geometrical progression, each succeeding quantity is equal to the product of the preceding quantity and some common factor; hence if a, b, and c be in geometrical progression,

$$\frac{b}{a}=\frac{c}{b};$$

$$\therefore\ b^2=ac,\ \text{and}\ b=\sqrt{ac}.$$

Hence the geometric mean between two quantities is equal to the square root of the product of the quantities.

In any geometrical progression the terms between the first and last terms are called the geometric means between the first and last. Thus, in the series 3, 9, 27, 81, 243 and 729, the numbers 9, 27, 81 and 243 are the four geometric means between 3 and 729.

Ex. 3. *Insert three geometric means between* 2 *and* 32.

To be able to insert a number of geometric means between any two numbers, we must know the value of the common ratio r. We have already seen that

$$r = \sqrt[n-1]{\frac{l}{a}}.$$

In our example n is 5, for there are 5 terms in the series, 2 extremes and 3 means;

$$\therefore\ r = \sqrt[n-1]{\frac{l}{a}} = \sqrt[4]{\frac{l}{a}} = \sqrt[4]{\frac{32}{2}} = \sqrt{\sqrt{16}} = 2;$$

$\therefore$ the series is 2, 4, 8, 16, 32, of which 4, 8 and 16 are the geometric means between 2 and 32.

To find the sum of n terms of any geometrical progression.

Let the series be a, ar, ar^2, ar^3, ... to n terms.

We have seen that $l = ar^{n-1}$; hence

$$lr = ar^{n-1} \times r = ar^{n-1+1} = ar^n.$$

Let s denote the sum of the series; then

$$s = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}.$$

Multiply both sides of this equation by r;

$$\begin{aligned} \text{then}\quad rs &= \phantom{a+{}} ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} + ar^n \\ \text{and}\quad s &= a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} \end{aligned}$$

by subtraction, $rs - s = ar^n - a$;

$$\text{that is,}\quad s(r-1) = a(r^n - 1);$$

$$\therefore\ s = \frac{a(r^n - 1)}{r - 1}.$$

We give an arithmetical illustration of this proof. Take the series 5, 15, 45, 135, 405, and denote the sum by s; then

$$s = 5 + (5 \times 3) + (5 \times 3^2) + (5 \times 3^3) + (5 \times 3^4).$$

Multiply both sides of this equation by the common ratio 3; then

$$\begin{aligned} 3s &= \phantom{5+{}} (5 \times 3) + (5 \times 3^2) + (5 \times 3^3) + (5 \times 3^4) + (5 \times 3^5) \\ s &= 5 + (5 \times 3) + (5 \times 3^2) + (5 \times 3^3) + (5 \times 3^4) \end{aligned}$$

$\therefore$ by subtraction,

$$2s = (5 \times 3^5) - 5 = 5(3^5 - 1);$$

$$\therefore\ s = \frac{5(3^5 - 1)}{2},$$

where 5 is the first term of the series, 3 the common ratio, the index 5 the number of terms in the series, and 2 the common ratio less 1. The result agrees exactly with the general result,

$$s=\frac{a(r^n-1)}{r-1}.$$

The value of a fraction is not altered by changing the signs of *all* the terms in the numerator and also of *all* the terms in the denominator, for this is equivalent to multiplying both numerator and denominator by -1. Hence we may express the value of s in two ways:

$$(1)\quad s=\frac{a(r^n-1)}{r-1},$$

$$(2)\quad s=\frac{a(1-r^n)}{1-r}.$$

It will be found convenient to use (2) in all cases except those in which r is positive and greater than 1.

For convenience of reference, we bring together the results now obtained:

$$l=ar^{n-1}, \quad\ldots\ldots\ldots\ldots\ldots(1)$$

$$s=\frac{a(r^n-1)}{r-1}, \quad\ldots\ldots\ldots\ldots\ldots(2)$$

$$s=\frac{a(1-r^n)}{1-r}. \quad\ldots\ldots\ldots\ldots\ldots(3)$$

Ex. 4. *A man buys 6 articles, paying 2d. for the first, 4d. for the second, 8d. for the third, and so on: find (1) the cost of the last article, and (2) the total cost of the articles.*

This is a geometrical series of 6 terms, the first of which is 2, and the common ratio 2; hence

$$l=ar^{n-1}=2\times 2^5=64,$$

$$\text{and}\quad s=\frac{a(r^n-1)}{r-1}=\frac{2(2^6-1)}{2-1}=126;$$

∴ cost of the last article = 64*d.* = 5*s.* 4*d.*
and total cost = 126*d.* = 10*s.* 6*d.*

Ex. 5. *Find the last term and also the sum of the series*

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\text{etc. } \textit{to 8 terms.}$$

$$r=\tfrac{1}{2},\quad n=8;$$

$$\therefore\ l=ar^{n-1}=1\times(\tfrac{1}{2})^7=\tfrac{1}{128},$$

$$\text{and}\quad s=\frac{a(1-r^n)}{1-r}=\frac{1\{1-(\frac{1}{2})^8\}}{1-\frac{1}{2}}=1\tfrac{127}{128}.$$

In a descending series r is a fraction, and a fraction becomes less and less in proportion to the number of times it is multiplied into itself. Thus,

$$\tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}; \quad \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{128}.$$

It follows, therefore, that in a descending series, if the number of terms is very large, r^n is very small, and that, if the number of terms is infinite, that is, without limit, r^n is infinitely small, practically 0; hence in an infinite series, that is, one containing an infinite number of terms, if r is a proper fraction, the value of r^n may be disregarded. Then

$$s = \frac{a}{1 - r}.$$

s here denotes the value towards which the series approaches as the number of its terms increases. Take the series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \text{etc.}$$

$$\begin{aligned}\text{The sum of two terms } &= 1 + \tfrac{1}{2} = 1\tfrac{1}{2},\\ \text{the sum of three terms} &= 1 + \tfrac{1}{2} + \tfrac{1}{4} = 1\tfrac{3}{4},\\ \text{the sum of four terms } &= 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = 1\tfrac{7}{8},\end{aligned}$$

the sum approaching nearer and nearer to 2 as the terms are added. It can never reach 2; but we can, by taking a sufficiently large number of terms, make it inconceivably less than 2, and therefore 2 is said to be the sum of the infinite series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \dots \text{ to infinity,}$$

for it is the limit towards which the sum of the series approaches.

In this series $a = 1$, $r = \tfrac{1}{2}$, and n is infinite;

$$\therefore\ s = \frac{a}{1 - r} = \frac{1}{1 - \frac{1}{2}} = 2,$$

which result must be understood in the sense explained above.

Ex. 6. *Sum to infinity* $(\tfrac{1}{20} + \tfrac{1}{30})$, $(\tfrac{1}{20} + \tfrac{1}{30})^2$, $(\tfrac{1}{20} + \tfrac{1}{30})^3$, *and so on.*

The series is $\tfrac{1}{12}$, $(\tfrac{1}{12})^2$, $(\tfrac{1}{12})^3$, and so on.

Here $r = \tfrac{1}{12}$, $a = \tfrac{1}{12}$, and n is infinite;

$$\therefore\ s = \frac{a}{1 - r} = \frac{\frac{1}{12}}{1 - \frac{1}{12}} = \frac{\frac{1}{12}}{\frac{11}{12}} = \frac{1}{11}.$$

A recurring decimal is an infinite geometrical series.

$$\cdot\dot{6}\dot{3} = 636363\ldots \text{ to infinity}$$

$$= \frac{63}{100} + \frac{63}{100^2} + \frac{63}{100^3} + \ldots \text{ to infinity.}$$

Here $r = \frac{1}{100}$, $a = \frac{63}{100}$, and n is infinite;

$$\therefore\ s = \frac{a}{1-r} = \frac{\frac{63}{100}}{1-\frac{1}{100}} = \frac{\frac{63}{100}}{\frac{99}{100}} = \frac{7}{11}.$$

EXAMPLES. XCV.

1. The first term is 4, the common ratio 3, and the number of terms 10: find the last term.

2. The first term is 5, the common ratio 4, and the number of terms 9: find the last term.

3. The last term is 6561, the common ratio 3, and the number of terms 7: find the first term.

4. The last term is 13122, the common ratio 3, and the number of terms 9: find the first term.

5. Find the eleventh term of the series $\frac{1}{64}$, $\frac{1}{16}$, $\frac{1}{4}$, $\cdots$, and so on.

6. The values of 15 things are in geometrical progression: the value of the first being 3*d.*, of the second 6*d.*, of the third 1*s.*, find the value of the last.

7. What is the 6th term of a geometrical progression the greater extreme of which is ·5 and the constant factor ·2?

8. Insert 5 geometric means between 3 and 2187.

9. Insert 3 geometric means between 1 and 81.

10. Insert 5 geometric means between $\frac{8}{125}$ and $\frac{125}{8}$.

11. Sum 5, 20, 80, and so on to 9 terms.

12. Sum 3, $4\frac{1}{2}$, $6\frac{3}{4}$, etc. to 5 terms.

13. Sum $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, etc. to infinity.

14. A farmer has 7 paddocks, the areas of which are in geometrical progression, the area of the first being 1 acre, of the second 4 acres, of the third 16 acres, and so on: how many acres are there in the 7 paddocks, and what is the area of the largest?

15. A man sells 6 horses, the first for £1, the second for £3, the third for £9, and so on: find how much he gets for the whole, and how much for the last?

CHAPTER XXXIV.

MENSURATION. SURFACES.

Illustrations and Definitions.

QUADRILATERALS.

A *quadrilateral* is any figure enclosed by four straight lines (Fig. 30).

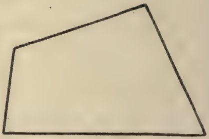


FIG. 30.

A *parallelogram* is a four-sided figure having its opposite sides parallel and equal and its opposite angles equal. When its angles are not right angles, as in the figure, it is called an *oblique parallelogram* (Fig. 31).

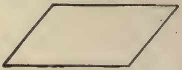


FIG. 31.

A *rectangle* is a parallelogram having each of its angles a right angle (Fig. 32). It would be sufficient to say having one of its angles a right angle. Why?

FIG. 32.

A *square* is a parallelogram having each of its angles a right angle (Fig. 33). Would it be sufficient to say having one of its angles a right angle? Why? A square is only a particular case of a rectangle. All its sides are equal.

FIG. 33.

A *rhombus* is a parallelogram having all its sides equal and its opposite angles equal, but its angles are not right angles (Fig. 34).

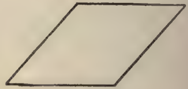


FIG. 34.

A *trapezium* is a four-sided figure having only two of its sides parallel (Fig. 35).

FIG. 35.

The *perimeter* of a figure is the sum of the lengths of its sides. It is proved by geometry that:

The diagonals of a parallelogram bisect each other.

Either diagonal of a parallelogram divides the parallelogram into two equal parts.

The diagonals of a rectangle are equal.

The diagonals of a square and of a rhombus intersect at right angles.

TRIANGLES.

An *equilateral triangle* is a triangle that has its three sides equal (Fig. 36).

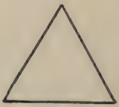


FIG. 36.

An *isosceles triangle* is a triangle that has two of its sides equal (Fig. 37).

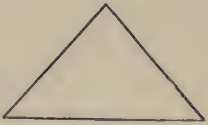


FIG. 37.

A *scalene triangle* is a triangle that has three unequal sides.

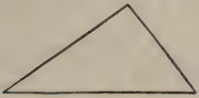


FIG. 38.

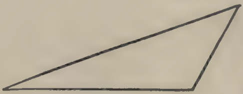


FIG. 39.

Other plane figures will be illustrated as we proceed.

The relation of the diagonal to the sides of a rectangle.

In the figure, $ABCD$ is the square on AB, and $BCEF$ is the square on the side BC; that is, they are the squares on the sides AB, BC of the right-angled triangle ABC. The side opposite the right angle of a right-angled triangle is called the *hypotenuse* of the triangle; hence AC is the hypotenuse of the triangle ABC.

The diagonals of equal squares are equal; hence the sides *AC*, *CF*, *FG*, *GA* of the figure *ACFG* are equal, and the angle *ACF* is a right angle, for the angles *BCA*, *BCF* are each half a right angle. Hence the figure *ACFG* is a square, and it is the square on the hypotenuse of the right-angled triangle *ABC*; and the four triangles of which it is composed are equal to the four triangles which make up the squares on the sides *AB*, *BC*. That is, the square on the hypotenuse of the right-angled triangle *ABC* is equal to the sum of the squares on the sides containing the right angle. This is only a particular illustration of a property of right-angled triangles that is known to be quite general. (Euclid, I. 47). We quote Euclid's Theorem: "*In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares on the other two sides.*"

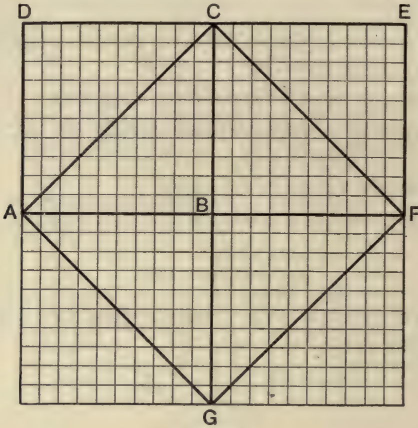


FIG. 40.

Let *ABC* be a right-angled triangle having the angle at *A* a right angle; and let a, b, c denote the units of length in *AB*, *AC*, and *BC* respectively. Then

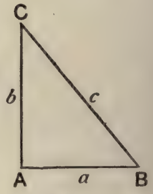


FIG. 41.

$$c^2 = a^2 + b^2; \quad \dots\dots(1)$$

and from this equation we derive the two following:

$$c^2 - a^2 = b^2, \quad \dots\dots(2)$$

$$c^2 - b^2 = a^2. \quad \dots\dots(3)$$

Hence, if any two sides of a right-angled triangle are known, it is easy to find the third side; for

$$(1) \qquad c^2 = a^2 + b^2;$$
$$\therefore\ c = \sqrt{a^2 + b^2}.$$
$$(2) \quad c^2 - a^2 = b^2;$$
$$\therefore\ \sqrt{c^2 - a^2} = b,$$

that is $\sqrt{(c-a)(c+a)} = b$;

and $(3) \quad c^2 - b^2 = a^2;$

$$\therefore\ \sqrt{c^2 - b^2} = a,$$

that is, $\sqrt{(c-b)(c+b)} = a.$

Given $a = 51$ yards, and $b = 68$ yards, to find c.

$$c^2 = (51)^2 + (68)^2$$
$$= 2601 + 4624 = 7225;$$
$$\therefore\ c = \sqrt{7225} = 85 \text{ yards}.$$

Ex. 1. *A triangular piece of ground has one of its angles a right angle; the side opposite the right angle is* 115 *yards long, and one of the sides containing the right angle is* 92 *yards long: find the length of the other side.*

Let x denote the length of the other side in yards; then

$$x^2 = (115)^2 - (92)^2$$
$$= (115 + 92)(115 - 92)$$
$$= 207 \times 23 = 4761;$$
$$\therefore\ x = \sqrt{4761} = 69 \text{ yards}.$$

Ex. 2. *The hypotenuse of a right-angled triangle is* 17 *feet, and the sum of the sides containing the right angle is* 23 *feet: find the sides.*

Let one side be x feet; then the other side is $(23 - x)$ feet;

then $x^2 + (23 - x)^2 = (17)^2,$

that is, $x^2 + 529 - 46x + x^2 = (17)^2;$

$$\therefore\ 2x^2 - 46x = (17)^2 - 529 = -240;$$
$$\therefore\ x^2 - 23x + \left(\tfrac{23}{2}\right)^2 = -120 + \left(\tfrac{23}{2}\right)^2$$
$$= \tfrac{49}{4};$$
$$\therefore\ x - \tfrac{23}{2} = \sqrt{\tfrac{49}{4}} = \pm\tfrac{7}{2},$$

that is, $x = \pm\tfrac{7}{2} + \tfrac{23}{2};$

$\therefore\ x = 15$ or 8, and the sides are 15 ft. and 8 ft.

The relation between the diagonal and the side of a square.

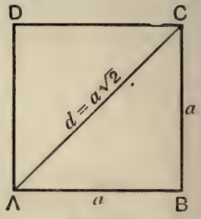


Fig. 42.

Let $ABCD$ be a square of which AC is a diagonal, and let each side contain a units of length, and the diagonal d of the same kind of unit. Then, because ABC is a right angle,

$$d^2 = a^2 + a^2 = 2a^2$$

(that is, twice the area of the square);

$$\therefore\ d = \sqrt{2a^2} = a\sqrt{2};$$

$$\therefore\ \text{diagonal} = \text{side} \times \sqrt{2}, \text{ ....}(1)$$

$$a\sqrt{2} = d;$$

$$\therefore\ a = \frac{d}{\sqrt{2}} = \frac{d\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \frac{d\sqrt{2}}{2},$$

$$\therefore\ \text{side} = \tfrac{1}{2}\text{ diagonal} \times \sqrt{2}. \text{ ....}(2)$$

The value of $\sqrt{2}$ should be memorised. To four decimal places, it is 1·4142. The log of $\sqrt{2}$ is ·1505.

Ex. 1. *The area of a square is* 200 *square feet: find the diagonal of the square.*

$$(\text{Diagonal})^2 = \text{twice the area}$$

$$= 2 \times 200;$$

$$\therefore\ \text{diagonal} = \sqrt{2\times 200} = \sqrt{20\times 20} = 20 \text{ ft.}$$

Ex. 2. *The side of a square is* 14 *poles: find to the nearest foot the length of the diagonal of the square.*

$$\text{Side} = 14 \text{ poles} = 231 \text{ ft.};$$

$$\text{diagonal} = \text{side} \times \sqrt{2}$$

$$= 231 \times \sqrt{2}.$$

Denote the diagonal by x; then

$$\log x = \log 231 + \log\sqrt{2}$$

$$= 2{\cdot}3636 + {\cdot}1505$$

$$= 2{\cdot}5141.$$

$$\text{Antilog} = 3267;$$

$$\therefore\ \text{diagonal} = 326{\cdot}7 \text{ ft.}$$

$$= 327 \text{ ft. nearly.}$$

By the methods of arithmetic:

$$231 \times \sqrt{2}$$

$$= 231 \times 1{\cdot}4142$$

$$= 326{\cdot}68$$

by contracted multiplication.

Ex. 3. *The diagonal of a square field is* 33 *chains* 24 *links: find the length of the side of the field.*

33 chains 24 links $= 33{\cdot}24$ chains;

$$\text{side} = \tfrac{1}{2}\,\text{diagonal} \times \sqrt{2}$$
$$= 16{\cdot}62 \text{ chains} \times \sqrt{2}.$$

Denote the length of the side by x; then

$$\log x = \log 16{\cdot}62 + \log \sqrt{2}$$
$$= 1{\cdot}2206 + {\cdot}1505$$
$$= 1{\cdot}3711.$$

$$\text{Antilog} = 2351;$$
$$\therefore\ x = 23{\cdot}51 \text{ chains}.$$

The answer is in excess by nearly 1 link, a trifle in a dimension of a large field.

$$16{\cdot}62 \times \sqrt{2}$$
$$= 16{\cdot}62 \times 1{\cdot}4142$$
$$= 23{\cdot}50\ldots$$

by contracted multiplication.

EXAMPLES. XCVI.

1. Draw on squared paper and to the scale of ·1 inch to a metre right-angled triangles having the sides containing the right angle of the following lengths, and in each case find by measurement and by calculation the length of the side opposite the right angle:

(*a*) 7 metres and 24 metres; (*b*) 16 metres and 63 metres; (*c*) 48 metres and 55 metres.

2. The hypotenuse of a right-angled triangle is 9·7 inches, and one of its sides is 6·5 inches: draw the triangle, and give the length of the third side.

3. A ship sails 36 miles due east and then 77 miles due north: draw on squared paper a plan (·1 inch to a mile) of the ship's course, and find by measurement and by calculation her distance from her starting point.

4. A town A lies due west of a town B, but the distance between them is unknown; a town C lies 45 miles due south of B and 53 miles from A: plot on squared paper the positions of the towns, and find by measurement and by calculation the distance of A from B.

5. Plot the points (0, 9) and (40, 0), join them by a straight line, and find by measurement and by calculation the length of the line.

6. A man travels 10 km. south, then 13 km. east, then 5 km. north, then 11 km. east, and finally 12 km. north: draw to scale ·1 inch to a km. a plan of his journey, and find by measurement and by calculation how far he now is from his starting point.

7. The diagonal of a square is 22 chains 18 links: find to the nearest link the length of the side of the square.

8. The diagonal of a rectangle is 125 chains, and one of the sides is 44 chains: find the other side, and the area of the rectangle.

9. The side of a square is 71 yards 2 feet: find to the nearest foot the length of the diagonal.

10. A steamer goes north for four hours at 13 knots per hour, and then east for 5 hours at 12 knots per hour; she now returns to port in a direct line and at the rate of 10 knots per hour: how long does she take to reach the port?

11. The hypotenuse of a right-angled triangle is 52 feet, and the sides containing the right angle are to each other as 5 to 12: find the two sides. Draw the triangle to any convenient scale, and show by measurement that the answer obtained by calculation is correct.

12. The hypotenuse of a right-angled triangle is 26 cm., and the sum of the sides containing the right angle is 34 cm.: find the lengths of the sides.

Geometrical Illustrations of Algebraical Identities.

I. Geometrical illustration of the identity

$$a(b+c+d+\dots)=ab+ac+ad+\dots.$$

Let $ABCD$ be a rectangle having a units in the side AD and $(b+c+d)$ units of length in the side AB,

then the rectangle $ABCD$

$=$ the rectangles $AF+EH+GC$

$=ab+ac+ad$

$=a(b+c+d)$.

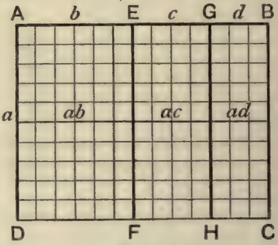


FIG. 43.

II. Geometrical illustration of the identity

$$(a+b)^2=a^2+2ab+b^2.$$

Let $ABCD$ be a square on the line AB, and let $AB=(a+b)$ units of length;

then $(a+b)^2=$ the square $ABCD$

$=$ the sq. GF

$+$ the sq. EK

$+$ the rect. AH

$+$ the rect. HC

$=a^2+b^2+2ab$;

that is, $a^2+2ab+b^2$.

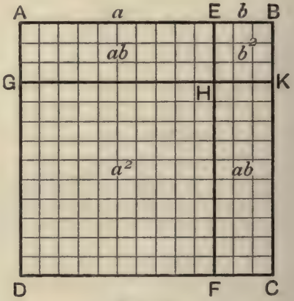


FIG. 44.

Hence a square of side $(a+b)$ units of length is larger than a square of a units of length by $(2ab+b^2)$ units of area.

The diagram (Fig. 44) may be used to illustrate the effect on the area of a rectangular figure when the linear dimensions are increased or decreased. (1) and (2) below apply only to squares, but (3) applies to any rectangle. In the diagram

(1) $AD = DG + 3$ units of length $= 13$ units of length,
$AB = GH + 3$ units of length $= 13$ units of length ;

$$\begin{aligned}\text{increase in area} &= (13)^2 - (10)^2\\ &= (13+10)(13-10)\\ &= 69 \text{ units of area.}\end{aligned}$$

(2)

$$\begin{aligned}\text{Increase in area} &= 2ab + b^2\\ &= 2 \times 10 \times 3 + 3^2\\ &= 69 \text{ units of area.}\end{aligned}$$

(3) DG is increased in length by $\frac{3}{10}$ of itself, and GH, that is AE, by $\frac{3}{10}$ of itself ;

$$\begin{aligned}\therefore \text{ area of enlarged figure} &= (1\tfrac{3}{10} \times 1\tfrac{3}{10}) \text{ of the original figure}\\ &= \tfrac{169}{100} \text{ of the original figure}\\ &= \text{the original} + \tfrac{69}{100} \text{ of the original figure}\\ &= (100 + 69) \text{ units of area ;}\end{aligned}$$

$$\therefore \text{ increase in area} = 69 \text{ units of area.}$$

The student should verify these results by computing the number of small squares. By working backwards with the diagram or by drawing another, he can show the effect on area of decrease in the linear dimensions of rectangular figures.

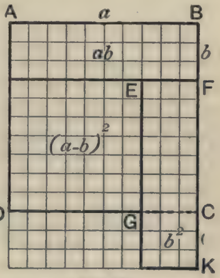


FIG 45.

III. Geometrical illustration of the identity $(a-b)^2 = a^2 - 2ab + b^2$.

Let $ABCD$ be a square on the line AB, and GK a square on the line CK drawn equal to BF.

Then $FK = AB$,

and rect. AF = rect. EK.

Hence $DE + 2AF = AC + GK$;

that is, $(a-b)^2 + 2ab = a^2 + b^2$;

$$\begin{aligned}(a-b)^2 &= a^2 + b^2 - 2ab\\ &= a^2 - 2ab + b^2.\end{aligned}$$

In the figure, $a = 10$ units of length, and $b = 3$ units of length;

$$\therefore\ (a-b)^2 = (10-3)^2 = 7^2 = 49 \text{ square units,}$$

and $a^2 - 2ab + b^2 = (10)^2 - 2 \times 10 \times 3 + 3^2$

$$= 100 - 60 + 9 = 49 \text{ sq. units.}$$

IV. Geometrical illustration of the identity

$$(a+b)(a-b) = a^2 - b^2.$$

In the following rectangular figures, let $a = 6$ and $b = 4$ units of length.

Then the rectangle $ABCD$ (Fig. 46)

$$= (a+b)(a-b)$$

$$= (6+4)(6-4) = 20 \text{ units of area,}$$

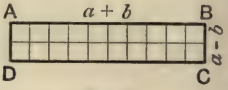


FIG. 46.

the square $ABCD$ (Fig. 47) $= a^2 = 6^2 = 36$ units of area,

and the square $EFGH$ (Fig. 47) $= b^2 = 4^2 = 16$ units of area.

Take the smaller from the larger square; then we have (Fig 47)

$$ABCD - EFGH = a^2 - b^2 = (36 - 16) \text{ units of area}$$

$$= 20 \text{ units of area;}$$

that is, $(a+b)(a-b) = a^2 - b^2$.

Verify by counting or by computing the number of small squares.

Fig. 47 will serve to illustrate the case of a rectangular space having another rectangle so placed within it as to leave a margin of uniform width between the two rectangles, as, for example, a path of uniform width running round a rectangle, etc. The area of the margin is obviously equal to the difference of the rectangles $ABCD$ and $EFGH$. Devise four other ways of finding the area of the space between these two rectangles. (The extension of the sides EF and HG and of the sides EH and FG will suggest two ways; and, the figures in this case being squares, the geometrical illustration of the identity $(a+b)^2 = a^2 + 2ab + b^2$ will suggest two others.)

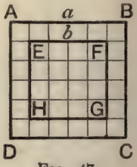


FIG. 47.

EXAMPLES. XCVII.

1. Four strips of carpet, each a units wide, have the following lengths respectively: b units, c units, d units, and e units; write down two expressions for the sum of their areas.

2. Find the increase in the area of the original rectangle in each of the following cases (verify on squared paper):

	Original rectangle.	Increase in length.
(*a*)	9 units by 9 units;	2 units in each side.
(*b*)	12 units by 12 units;	3 units in each side.
(*c*)	16 units by 8 units;	4 units in each side.
(*d*)	10 units by 8 units;	1 unit in each side.
(*e*)	12 units by 7 units;	1st side by 4, 2nd side by 3 units.
(*f*)	9 units by 13 units;	1st side by 2, 2nd side by 5 units.

3. Find the decrease in the area of the original rectangle in each of the following cases (verify on squared paper):

	Original rectangle.	Decrease in length.
(*a*)	8 units by 8 units;	2 units in each side.
(*b*)	13 units by 13 units;	3 units in each side.
(*c*)	14 units by 10 units;	4 units in each side.
(*d*)	16 units by 11 units;	5 units in each side.
(*e*)	17 units by 15 units;	1st side by 5, 2nd side by 4 units.
(*f*)	12 units by 16 units;	1st side by 4, 2nd side by 3 units.

4. Each side of a rectangle is increased by $\frac{2}{5}$ of itself, and the corresponding increase in area is 72 units of square measure: find the area of the original rectangle, and give in whole numbers two sets of dimensions that answer to it.

5. A rectangular flower bed, 16 feet long and 12 feet wide, is surrounded by a path 4 feet wide: find the area of the path.

6. Draw on squared paper rectangles, the sum of the adjacent sides of which is 12 units, and observe the changes in the length and breadth, and in the corresponding areas, of the figures. Which of the figures has the largest area, and which the smallest? What inference do you draw from these results?

7. Draw on squared paper a square of 64 units of area, and three other rectangles having the same area as the square: compare the lengths of the perimeters of the figures, and state what inference you draw from these results.

The Rectangle further considered.

We have seen that, when the length and breadth are expressed in the same kind of unit, the area of a rectangle and its sides have the following relations:

$$\text{length} \times \text{breadth} = \text{area},$$

$$\text{breadth} = \frac{\text{area}}{\text{length}},$$

$$\text{length} = \frac{\text{area}}{\text{breadth}}.$$

Denoting area by A, length by l, and breadth by b, we have

$$l \times b = A, \quad \text{.............................}(1)$$

$$b = \frac{A}{l}, \quad \text{.............................}(2)$$

$$l = \frac{A}{b}, \quad \text{.............................}(3)$$

If the rectangle is a square, length and breadth are equal; then

$$l^2 = A \quad \text{.............................}(1)$$

$$\text{and} \quad l = \sqrt{A}. \quad \text{.............................}(2)$$

It must be carefully remembered that in these equations A denotes the number of square units of the denomination corresponding to the linear units denoted by l and b.

The number of examples already given renders it unnecessary to work any more by arithmetical methods, but we may show the application of logarithms and algebra to the solution of questions on rectangular figures.

Ex. 1. *Find the area of a rectangular field* 18 *chains* 25 *links long by* 13 *chains* 47 *links wide.*

$$\begin{aligned} A &= l \times b \\ &= (18{\cdot}25 \times 13{\cdot}47) \text{ sq. chains;} \\ \therefore \log A &= \log 18{\cdot}25 + \log 13{\cdot}47 \\ &= 1{\cdot}2613 + 1{\cdot}1294 \\ &= 2{\cdot}3907. \\ \text{antilog} &= 2459; \\ \therefore A &= 245{\cdot}9 \text{ sq. chains.} \end{aligned}$$

If the student will check by arithmetical methods, he will find that this result is too large by ·0725 of a square chain, a very small error in a field of over 24 acres.

Ex. 2. *The area of a rectangular orchard is* 6·22 *acres and one of its sides is* 9·75 *chains: find the other side.*

Let x denote the other side in chains; then

$$\begin{aligned} x &= \frac{62{\cdot}2}{9{\cdot}75} \text{ chains;} \\ \therefore \log x &= \log 62{\cdot}2 - \log 9{\cdot}75 \\ &= 1{\cdot}7938 - {\cdot}9890 \\ &= {\cdot}8048. \\ \text{antilog} &= 6380; \\ \therefore x &= 6{\cdot}38 \text{ chains.} \end{aligned}$$

Ex. 3. *The perimeter of a rectangular plot of ground is* 120 *yards and the area of the plot is* 875 *square yards: find the length and the breadth of the plot.*

$$\begin{aligned}\text{Sum of length and breadth} &= \tfrac{1}{2}\text{ of perimeter}\\ &= \tfrac{1}{2}\text{ of } 120 \text{ yards}\\ &= 60 \text{ yards.}\end{aligned}$$

Let x yards be the breadth;

$$\text{then}\quad (60-x) = \text{the length};$$
$$\therefore\ \text{area} = x(60-x)\text{ sq. yards};$$
$$\therefore\ x^2-60x = -875;$$
$$\therefore\ x^2-60x+(30)^2 = -875+(30)^2;$$
$$\therefore\ x-30 = \pm 5;$$
$$\therefore\ x = 25 \text{ or } 35.$$

35 is inadmissible, for it would make the width larger than the breadth;

hence length $=35$ and breadth $=25$ yards.

Ex. 4. *The area of a rectangular field is* 42 *acres and the diagonal is* 37 *chains: find the length and breadth.*

$$42 \text{ acres} = 420 \text{ sq. chains.}$$

Denote the sides by a and b; then

$$a^2+b^2 = (37)^2$$
$$\text{and}\quad ab = 420 \text{ sq. chains};$$
$$\therefore\ 2ab = 840 \text{ sq. chains};$$
$$\therefore\ a^2+2ab+b^2 = (37)^2+840 = 2209;$$
$$a+b = \sqrt{2209} = 47;$$
$$\therefore\ a = 47-b.$$

Putting $47-b$ for a in ab, we have

$$ab = b(47-b);$$
$$\therefore\ b(47-b) = 420 \text{ sq. chains};$$
$$\therefore\ b^2-47b+\left(\tfrac{47}{2}\right)^2 = -420+\left(\tfrac{47}{2}\right)^2.$$

Solving this equation, we have

sides $=35$ and 12 chains respectively.

EXAMPLES. XCVIII.

1. Find the area of a square field the side of which is 9 chains 17 links.

2. A road is 73·7 chains long and $1\frac{1}{2}$ chains wide: find its area.

3. A rectangular field contains 92·66 acres and its length is 33 chains 60 links: find its breadth.

4. A piece of paper is ·815 m. long and ·659 m. wide: find its area in square centimetres.

5. The perimeter of a rectangular piece of ground is 238 metres and the area 3120 square metres: find the length and breadth.

6. The perimeter of a rectangular piece of ground is 206 feet and the area 293 square yards 3 square feet: find the length and breadth of it.

7. The area of a rectangular farm is 277·2 acres and the diagonal 85 chains: find the length and breadth of the farm.

8. The area of a garden is 341 square yards 3 square feet and the diagonal is 80 feet: find the length of the fence that encloses the garden.

Papering and Carpeting Rooms.

The student has only to examine any rectangular room to see that it has two pairs of corresponding walls, namely, two sides of the same size and shape and two ends of the same size and shape, and that the floor and the ceiling are also of the same size and shape. Hence, denoting the number of units in the length by l, the number in the breadth by b, and the number in the height by h, we have

$$\text{circuit of room} = 2l + 2b = 2(l+b) \text{ linear units,} \quad \dots\dots\dots(1)$$

$$\text{area of walls} = 2(l+b)h \text{ units of area,} \quad \dots\dots\dots(2)$$

$$\text{area of floor} = lb \text{ units of area,} \quad \dots\dots\dots(3)$$

$$\text{area of floor and ceiling} = 2lb \text{ units of area.} \quad \dots\dots\dots(4)$$

It is obvious that the area of a floor is equal to the area of the strip of carpet required to cover the floor, and that the area of the walls of a room is equal to the area of the paper required to cover the walls.

In all cases of rectangles $\frac{A}{b} = l$;

$$\therefore \frac{\text{area of floor}}{\text{breadth of strip of carpet}} = \text{length of strip of carpet required to cover the floor}$$

and $$\frac{\text{area of walls}}{\text{breadth of piece of paper}} = \text{length of paper required to cover the walls.}$$

Ex. 1. *The length of the side of a square room is* 15 *feet: how much carpet* $2\frac{1}{2}$ *feet wide is required to cover the floor?*

$$\text{Area of floor} = (15 \times 15) \text{ square feet,}$$

$$\text{width of carpet} = 2\tfrac{1}{2} \text{ feet;}$$

$$\therefore \text{ length of carpet required} = \frac{15 \times 15}{2\frac{1}{2}} \text{ feet}$$

$$= 90 \text{ ft.} = 30 \text{ yards.}$$

This result is only theoretically correct, for in the working we have disregarded the waste caused by adjusting the pattern of the carpet.

Ex. 2. *A room is* 23 *ft.* 8 *ins. long,* 15 *ft.* 10 *ins. wide, and* 11 *ft.* 9 *ins. high; it has two windows each* $9\frac{1}{2}$ *ft. high by* 5 *ft. wide, a fire place* 4 *ft.* 6 *ins. high by* 6 *ft. wide, and a door* 7 *ft.* 6 *ins. high by* 3 *ft.* 6 *ins. wide: find the cost of papering it at* 35 *shillings per piece of* 12 *yds., the width of the paper being* 2 *ft.* 2 *ins.*

$$\text{Circuit of room} = 2(23 \text{ ft. } 8 \text{ in.} + 15 \text{ ft. } 10 \text{ in.})$$

$$= 79 \text{ feet.}$$

$$\text{Height of room} = 11\tfrac{3}{4} \text{ ft.};$$

$$\therefore \text{ area of walls} = \left(79 \times 11\tfrac{3}{4}\right) \text{ sq. ft.} = 928\tfrac{1}{4} \text{ sq. ft.}$$

$$\text{Area of windows} = \left(9\tfrac{1}{2} \times 5 \times 2\right) \text{ sq. ft.} = 95 \text{ sq. ft.},$$

$$\text{area of fire place} = \left(4\tfrac{1}{2} \times 6\right) \text{ sq. ft.} = 27 \text{ sq. ft.},$$

$$\text{area of door} = \left(7\tfrac{1}{2} \times 3\tfrac{1}{2}\right) \text{ sq. ft.} = 26\tfrac{1}{4} \text{ sq. ft.};$$

$$\therefore \text{ area not to be papered} = \left(95 + 27 + 26\tfrac{1}{4}\right) \text{ sq. ft.} = 148\tfrac{1}{4} \text{ sq. ft.};$$

$$\therefore \text{ area to be papered} = \left(928\tfrac{1}{4} - 148\tfrac{1}{4}\right) \text{ sq. ft.} = 780 \text{ sq. ft.}$$

$$\text{Width of paper} = 2\tfrac{1}{6} \text{ ft.};$$

$$\therefore \text{ length of paper required} = \frac{780}{2\frac{1}{6}} \text{ ft.} = 360 \text{ ft.} = 120 \text{ yards};$$

$$\therefore \text{ number of pieces of paper required} = \tfrac{120}{12} = 10;$$

$$\therefore \text{ cost of paper} = (35 \times 10) \text{ shillings} = £17.\ 10s.$$

Again the result is only theoretically correct; for in the working no account is taken of the adjustment of the pattern. The room would certainly require at least a part of another piece of paper, and therefore the whole piece would be charged for. In the examples given below to be worked by the student, no account is taken of these circumstances.

Ex. 3. *The cost of carpeting a room whose length is twice its breadth, at* 5*s. a square yard, was* £6. 2*s.* 6*d., and the painting of the walls at* 9*d. a square yard was* £2. 12*s.* 6*d.: find the height of the room.*

$$\text{Cost of carpet} = £6.\ 2s.\ 6d = 122\tfrac{1}{2} \text{ shillings},$$

$$\text{cost per yard of carpet} = 5 \text{ shillings};$$

$$\therefore \text{ area of floor} = \frac{122\frac{1}{2}}{5} = \frac{49}{2} \text{ sq. yards.}$$

$$l = 2b;$$

$$\therefore 2b^2 = \text{area of room} = \tfrac{49}{2} \text{ sq. yds.};$$

$$\therefore\ b^2 = \frac{49}{4} \text{ sq. yds.};$$

$$\therefore\ b = \sqrt{\frac{49}{4}} = \frac{7}{2} \text{ yards} = 10\tfrac{1}{2} \text{ ft.};$$

$$\therefore\ \text{length} = \left(10\tfrac{1}{2} \times 2\right) \text{ ft.} = 21 \text{ feet.}$$

Cost of painting the walls = £2. 12*s.* 6*d.* = 630*d.*,

cost per sq. yard of painting = 9*d.*;

$$\therefore\ \text{area of walls} = \frac{630}{9} = 70 \text{ sq. yds.} = 630 \text{ ft.}$$

$$\text{Circuit of walls} = 2\left(21 + 10\tfrac{1}{2}\right) \text{ ft.} = 63 \text{ ft.};$$

$$\therefore\ \text{height of walls} = \frac{630}{63} \text{ ft.} = 10 \text{ ft.}$$

EXAMPLES. XCIX.

1. Find the cost of carpeting a floor 120 ft. long by 80 ft. wide, with carpet 1½ yd. wide, at 4*s.* 6*d.* per yd.

2. Find the cost of carpeting a room 15 ft. 4 ins. by 10 ft. 6 ins., with carpet 2 ft. 4 ins. wide, at 3*s.* 6*d.* per yd.

3. How many boards 11 ft. by 13 ins. will be required to floor a room 26 ft. by 27 ft. 6 ins.?

4. A room is 35 ft. by 25 ft. and its floor is laid with boards 17 ft. 6 ins. by 15 ins.: find the number of boards and their cost at 13*s.* 6*d.* per sq. yd.

5. How many tiles each 7 ins. square will be required to cover a floor 19 ft. 3 ins. by 13 ft. 5 ins.?

6. A path 8 ft. wide, surrounding a rectangular court 60 ft. long and 36 ft. wide, is to be paved with tiles 9 ins. by 4 ins.: how many tiles will be required?

7. The area of a rectangular enclosure being 33 sq. po. 1 yd. 6 ft. 108 ins. and the length 9 po. 1 ft. 6 ins., what is the width?

8. The area of a field is 7 ac. 1 ro. 15 po. and its length 453 yds. 2 ft. 3 ins.: find its width.

9. What length must be cut off from a piece of land 110 yds. wide that the area of the part cut off may be 2⅝ ac.?

10. What will it cost to paper a room 21 ft. 5 ins. long, 18 ft. 7 ins. broad, and 10 ft. high, with paper 25 ins. wide at 7*d.* per ft.?

11. Find the cost of papering a room 28 ft. 6 ins. long, 18 ft. 9 ins. broad, and 12 ft. high, with paper 1 ft. 9 ins. wide at 2*s.* 6*d.* per yd.

12. A room measures 16 ft. by 21 ft. and is 11 ft. high, in it are two windows each 8 ft. by 4 ft. and a door 7 ft. by 3 ft.: find the cost of papering it with paper 2 ft. wide at 2½*d.* per yd.

13. How much paper ¾ yd. wide will be required for a room 22 ft. long, 14 ft. wide, and 9 ft. high, there being two windows and two doors, each 6 ft. by 3 ft.?

14. A room is 22¾ ft. long, 20 ft. 10 ins. wide, and 18 ft. 5 ins. high: allowing for a door 7 ft. 6 ins. by 3 ft. 4 ins., and two windows each 5 ft. 2 ins. by 3 ft. 6 ins., find to the nearest penny the cost of papering it at 3*s.* 4½*d.* per sq. yd.

15. Two rectangular rooms of the same height were papered; the first was 16 ft. long and 14 ft. wide, and the second 14 ft. long and 12 ft. wide; the paper was 1 ft. 9 ins. wide and 2*s.* 9*d.* per piece of 12 yds., and the hanging cost 9*d.* per piece; the whole expense was 4 guineas: find the height of the rooms.

16. A grass plot is 189 ft. long and 144 ft. wide, and it contains 5 square flower beds, the sides of which are 10 ft.: find the number of square yds. of grass.

17. A rectangular court has a path of the uniform width of 3 yds. 1 ft. running round it; the length of the court (including the path) is 40 yds. and the breadth 30 yds.: find the cost of paving the path at 4*s.* 6*d.* per sq. yd. and covering the remainder of the court with turf at 13 shillings per 200 sq. ft.

18. The area of a square cricket field is 9 ac. 3 ro. 8·16 poles; a path of the uniform width of 3·9 yds. is made close to the boundary (on the inner side) of the field at a cost of 4*d.* per square yard, and the remainder of the field is laid down in turf at a cost of 5*s.* 6*d.* per hundred square yards: find to the nearest penny the total cost of preparing the field.

19. A rectangular grass plot measures 320 yards by 160 yards; all round it (on the outer side of the boundary) is a gravel path 6 ft. broad; the price of making the grass plot is 6*d.* per square yard: what must be the price of the gravel path per square yard that the path may cost £1183. 4*s.* less than the grass plot?

20. It costs £643. 10*s.* 9*d.* to level and turf a square cricket ground at 9*d.* per square yard: find the cost of inclosing it with an iron railing at 7*s.* 6*d.* per yard.

21. The cost of carpeting a room, whose length is twice its breadth, at 7*s.* 6*d.* per square yard, was £18. 15*s.*: find the dimensions of the room.

22. The cost of paving a court-yard 3½ times as long as it is broad, at 5*s.* 6*d.* per square yard, was £15. 8*s.*: find the dimensions of the court.

23. The area of a rectangular field is 39 ac. 32 poles and one side is double the other: find the sides.

The relation between the rectangle and the oblique parallelogram.

The following figure shows a rectangle *ABCD* and an oblique parallelogram *BECD* on the same base *CD* and between the same parallels *AE* and *CD*; that is, two parallelograms having the same base and the same *vertical height* or *altitude.*

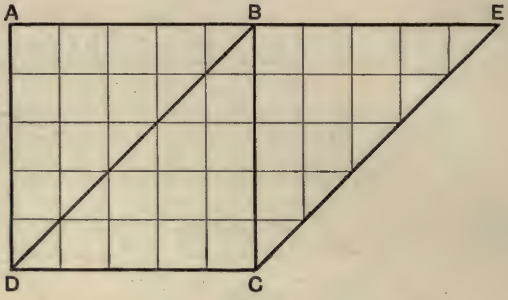


FIG. 48.

By counting the small squares, we see

(1) that the parallelograms are equal in area, and

(2) that the triangle *BCD* has half the area of either of the parallelograms.

The figure illustrates what geometry proves to be universally true, namely,

(*a*) *that parallelograms on the same base and between the same parallels* (i.e. *of the same altitude*) *are equal in area;*

(*b*) *that a triangle is half any parallelogram on the same base and between the same parallels* (i.e. *of the same altitude*).

Hence (1) *area of any parallelogram* = *base* × *altitude*

and (2) *area of any triangle* = $\frac{1}{2}$(*base* × *altitude*),

where base is put for the number of linear units in the base and altitude for the number of linear units in the altitude.

The student will readily see how to derive other equations from (1) and (2).

NOTE. Where *height* is used in the examples, *vertical height* or *altitude* is meant.

EXAMPLES. C.

1. Find, in acres and poles, the area of a triangular field the base of which is 12 ch. 25 lks. and height 8 ch. 50 lks.

2. The base of a triangular plot of ground is 16·75 ft. and its height 6·24 ft.; find its area.

3. The area of a triangular field is 14 acres and its altitude is 175 yds.: find the length of its base.

4. The area of a triangular field is 7 acres 3 roods 14 poles and the length of the base is $16\frac{1}{2}$ chains: find the perpendicular.

5. What is the rent of a triangular field the base of which is 33·3 chains and the perpendicular 22·2 chains, at £1. 15*s.* per acre? (Answer to the nearest penny.)

6. The base of a triangle is 9 ft. 8 inches and the altitude 6 ft. 4 inches: find its area.

7. Find the area of a triangular garden the base of which is 25 poles and height 18 poles.

8. The base of a triangular field is 75 poles and the altitude 57 poles: find the area of the field.

9. The area of a triangular field is 1737 acres and the altitude is 144 chains: find the base.

10. A rectangular park, one side of which is twice as long as the other, contains 500 acres: how much ground will be occupied by a road 15 ft. wide running round immediately inside the boundary of it?

11. A rectangular field containing 9 ac. 1 ro. $16\frac{1}{2}$ po. is one-third as broad as it is long: find the distance in poles round it.

12. The sides of a rectangular plot of ground are 16 ft. and 24 ft.: find how much border must be taken off all round that the area left may be 240 sq. ft.

13. The length of a rectangular garden is to its breadth as 6 to 5; one-sixth of the garden is planted and the remainder, 625 sq. yards, remains to be planted: find the length of the garden.

Given the three sides of a triangle, to find the area.

Ex. 1. *Find the area of a triangle the sides of which are* 12 *cm.*, 17 *cm.*, *and* 25 *cm.*

Let ABC represent the triangle. Draw AD perpendicular to BC; then ADB and ADC are right-angled triangles.

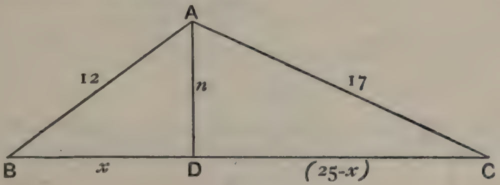


FIG. 49.

Let n denote the number of units in AD and x the number in BD; then

$$DC = (25 - x) \text{ units};$$

$$\therefore\ n^2 = (12)^2 - x^2, \text{ and also } (17)^2 - (25 - x)^2;$$

$$\therefore\ (12)^2 - x^2 = (17)^2 - (25 - x)^2.$$

Solving this equation, we have $x=\frac{48}{5}$;

$$\therefore\ n^2=(12)^2-(\tfrac{48}{5})^2,$$

the solution of which gives

$$n=7\tfrac{1}{5};$$

$$\therefore\ \text{area of triangle } ABC=\tfrac{1}{2}(25\times7\tfrac{1}{5})\text{ sq. cm.}$$
$$=90\text{ sq. cm.}$$

We have adopted this method to show the application of algebra to the solution of such questions. We shall now give the usual method of finding the areas of triangles the sides of which are of unequal lengths.

We know from Trigonometry that the area of a triangle in terms of its sides is expressed by the formula

$$\text{area of triangle}=\sqrt{s(s-a)(s-b)(s-c)},$$

where s represents $\frac{1}{2}$ the sum of the lengths of the sides and a, b, c the lengths of the sides of the triangle; that is,

$$s=\tfrac{1}{2}(a+b+c).$$

Hence in our triangle $s=\frac{1}{2}(25+17+12)=27.$

$$s-a=27-25=2,$$
$$s-b=27-17=10,$$
$$s-c=27-12=15;$$

$$\begin{aligned}\therefore\ \text{area of triangle}&=\sqrt{27\times2\times10\times15}\\&=\sqrt{9\times3\times2\times2\times5\times3\times5}\\&=\sqrt{3^2\times3^2\times2^2\times5^2}\\&=3\times3\times2\times5\\&=90\text{ sq. cm.}\end{aligned}$$

When the area of any triangle has been obtained, it is easy to find the vertical distance from any vertex to the side opposite the vertex; for

$$\text{altitude}=\frac{\text{twice area}}{\text{base}}=\frac{90\times2}{25}\text{ (in the present triangle)}$$

$$=7\tfrac{1}{5}\text{ cm., the result we obtained by another method.}$$

We shall now work an example by logarithms.

Ex. 2. *Find the area of a triangle the sides of which are* 95 *yards,* 168 *yards, and* 193 *yards.*

Let x denote the area; then

$$\begin{aligned} x &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{228 \times 133 \times 60 \times 35}; \\ \therefore \log x &= \tfrac{1}{2} \log (228 \times 133 \times 60 \times 35) \\ &= \tfrac{1}{2}(2{\cdot}3579 + 2{\cdot}1239 + 1{\cdot}7782 + 1{\cdot}5441) \\ &= 3{\cdot}9020. \end{aligned}$$

antilog = 7980;

$\therefore$ area of triangle = 7980 sq. yards.

It is usual to put Δ for the area of a triangle. By means of the formula,

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},$$

we can find the area of any triangle, but it is frequently convenient to be able to find the area of an equilateral triangle in another way.

Let ABC be an equilateral triangle, and let the perpendicular AD be drawn from the vertex A on the side BC.

Then the triangles ADB and ADC are equal in every respect (Euclid I. 26). Hence $BD = DC$ and $BC = 2BD$.

Denote BD by a units of length; then $AB = 2a$ units.

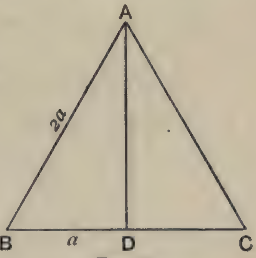


FIG. 50.

$$AD^2 = (2a)^2 - a^2 = 4a^2 - a^2 = 3a^2;$$

$$\therefore \ AD = \sqrt{3a^2} = a\sqrt{3} = \tfrac{1}{2} \text{ side} \times \sqrt{3}.$$

$$\begin{aligned} \text{Area of triangle } ABC &= \tfrac{1}{2}(AD \times BC) \\ &= \tfrac{1}{2}(a\sqrt{3} \times 2a) \\ &= a^2\sqrt{3} \\ &= (\tfrac{1}{2} \text{ side})^2 \times \sqrt{3} \\ &= \tfrac{1}{4}(\text{side}^2 \times \sqrt{3}). \end{aligned}$$

Hence the area of an equilateral triangle $= \dfrac{\text{side}^2 \times \sqrt{3}}{4}$.

The value of $\sqrt{3}$ to five decimal places is 1·73205.

Ex. 3. *Find the area of an equilateral triangle the sides of which are 39 yards long.*

Let x denote the area; then

$$x = \frac{\text{side}^2 \times \sqrt{3}}{4}$$

$$= \frac{(39)^2 \times \sqrt{3}}{4};$$

$$\therefore \log x = 2 \log 39 + \tfrac{1}{2} \log 3 - \log 4$$

$$= 3{\cdot}4207 - {\cdot}6021$$

$$= 2{\cdot}8186.$$

$$\text{antilog} = 6586;$$

$$\therefore \text{ area of triangle} = 658{\cdot}6 \text{ sq. yards.}$$

Note. When the sides of a triangle are lettered a, b, c, it is usual to assign a to the side opposite the angle at A, b to the side opposite the angle at B, and c to the side opposite the angle at C.

EXAMPLES. CI.

1. Find the areas of the triangles the three sides of which are respectively:

(*a*) 33, 34, and 65 yards. (*b*) 15, 37, and 44 yards.

(*c*) 25, 33, and 52 yards. (*d*) 20, 37, and 51 metres.

2. In the triangle ABC the side a is 15 inches, the side b 13 inches, and the side c 4 inches: draw the triangle to the scale of ·1 inch to an inch, find the length of the perpendicular from A to BC, and estimate the area of the triangle (1) from this result and (2) by means of the formula for the area of a triangle when the three sides are given.

3. In the triangle ABC the side a is 48 yards, the side b 29 yards, and the side c 35 yards: draw the triangle to the scale of ·1 inch to a yard, find the length of the perpendicular from C to AB, and estimate the area of the triangle (1) from this result and (2) from the formula for the area of a triangle when the three sides are given.

4. Each side of a triangular field is 10 chains: find the area of the field in acres, correct to two decimal places. (Draw on squared paper a plan of the field to the scale of ·1 inch to a chain, and test your answer by counting the squares and parts of squares.)

5. On squared paper, draw to any suitable scale a plan of a triangular field each side of which is 25 chains, and find in acres the area of the field.

6. Draw to any convenient scale an isosceles triangle the base of which is 12 feet and each side 10 feet, and find the area of the triangle.

7. A field is in form an equilateral triangle and the fence that bounds it is 51 chains: draw the field to any convenient scale and find its area in acres.

8. A flower plot is in form an isosceles triangle the base of which is 12 feet and the perimeter 44 feet: draw the plot to any convenient scale and find its area.

The Rhombus and the Trapezium.

Let $ABCD$ be a rhombus: then, because the diagonals bisect each other at right angles, we have

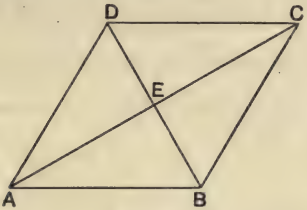


FIG. 51.

$$\begin{aligned}\text{area of rhombus } ABCD &= \text{twice the area of triangle } ABC\\ &= AC \times EB\\ &= \tfrac{1}{2}(AC \times BD).\end{aligned}$$

Hence the area of a rhombus is equal to one half the product of its diagonals.

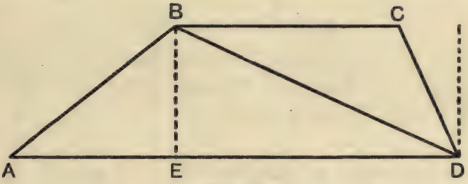


FIG. 52.

Let $ABCD$ be a trapezium having the sides AD, BC parallel; then the dotted line BE is the altitude of each of the triangles into which the rhombus is divided, for it is perpendicular to the parallel sides AD, BC. Hence we have

$$\begin{aligned}\text{area of } ABCD &= \text{area of triangle } ABD + \text{area of triangle } BCD\\ &= \tfrac{1}{2}(AD \times BE) + \tfrac{1}{2}(BC \times BE)\\ &= \tfrac{1}{2}(AD + BC) \times BE\\ &= \tfrac{1}{2} \text{ sum of parallel sides} \times \text{height}.\end{aligned}$$

Hence, to find the area of a trapezium, multiply half the sum of the parallel sides by the perpendicular distance between them.

Hence the area of a trapezium

$= \frac{1}{2}$ (*the sum of the parallel sides* × *height*).

EXAMPLES. CII.

1. The diagonals of a rhombus are 18 yds. and 16 yds. respectively: find the area.

2. The diagonals of a rhombus are 25 yds. and 35 yds. respectively: find the area.

3. The diagonals of a rhombus are 21 chains and 44 chains respectively: find the area.

4. The area of a rhombus is 990 sq. yds. and one of its diagonals 55 yds.: find the other diagonal.

5. The area of a rhombus is 130·5 acres and one of its diagonals is 45 chains: find the other diagonal.

6. The parallel sides of a quadrilateral are 18 yds. and 12 yds. respectively, and the height of it is 16 yds.: find the area.

7. The parallel sides of a quadrilateral are 32 chains and 18 chains respectively, and the height of it is 30 chains: find the area.

8. The area of a quadrilateral with two of its sides parallel is 5 sq. chains, and the sum of the parallel sides is 625 links: find the perpendicular distance between the parallel sides.

9. The area of a quadrilateral with two of its sides parallel is 14·7 sq. chains, and the sum of the parallel sides is 1225 links: find the perpendicular distance between the parallel sides.

Any Quadrilateral.

By drawing a diagonal we can divide any quadrilateral into two triangles, and, if sufficient data are given, find the area of the quadrilateral from the areas of the two triangles.

Let *ABCD* represent a quadrilateral field of which the following measurements have been taken: $AB = 16$ ch., $BC = 19$ ch., $CD = 27$ ch., $DA = 13$ ch., and the diagonal $AC = 24$ ch.

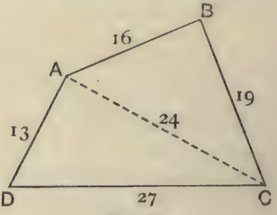


FIG. 53.

Area of $ABCD$ = sum of areas of triangles *ACD* and *ABC*, and the area of the triangles can be found from the formula

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)},$$

$$\triangle ACD = \sqrt{32 \times 8 \times 5 \times 19} = 156 \text{ sq. ch.},$$

$$\triangle ABC = \sqrt{29{\cdot}5 \times 13{\cdot}5 \times 10{\cdot}5 \times 5{\cdot}5} = 151{\cdot}7 \text{ sq. ch.};$$

$$\therefore \text{ area of } ABCD = 307{\cdot}7 \text{ sq. ch.}$$

$$= 30{\cdot}8 \text{ acres (nearly).}$$

The working is done by means of logarithms.

In practice it is often found more convenient to find the area of a quadrilateral by means of a diagonal and perpendiculars drawn to it from the opposite angles, as in the accompanying figure, in which AC is a diagonal, and BE, DF perpendiculars drawn to it from B and D respectively.

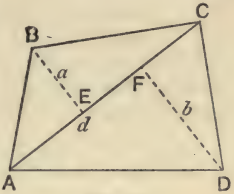


FIG. 54.

Denote AC by d, BE by a, and DF by b; then

$$\begin{aligned} \text{area of } ABCD &= \text{area of } (ABC + ADC) \\ &= \tfrac{1}{2}da + \tfrac{1}{2}db \\ &= \tfrac{1}{2}d(a+b) \\ &= \tfrac{1}{2} \text{ diagonal} \times \text{sum of perpendiculars drawn to it from the opposite angles.} \end{aligned}$$

Such perpendiculars are called *offsets*;

hence area of a quadrilateral $= \frac{1}{2}$ diagonal × (sum of offsets).

To find the area of an irregular figure like the following:

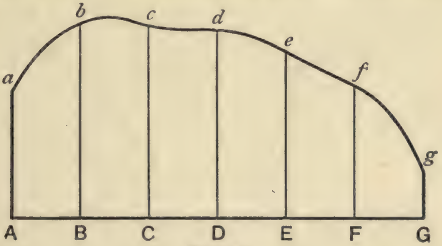


FIG. 55.

Divide AG into any *even* number of equal parts AB, BC, CD, ... From the points of division draw straight lines (called *ordinates*) Aa, Bb, ... at right angles to AG to meet the curve adg. Then we can find the approximate area of the figure as follows:

Add together the first and last ordinates, four times the sum of the even ordinates, and twice the sum of the odd ordinates (omitting the first and last); then multiply the result by one-third of the distance between any two consecutive ordinates.

Let A denote the sum of the first and last ordinates, B the sum of the even ordinates, C the sum of the odd ordinates other than the first and the last, and s the space between any two consecutive ordinates; then

$$\text{area of the figure} = \frac{s}{3}(A + 4B + 2C) \text{ units of area.}$$

When the curved side extends to the ends of the straight side as in the adjoining figure, the formula becomes

$$\text{area} = \frac{s}{3}(0 + 4B + 2C) \text{ units of area.}$$

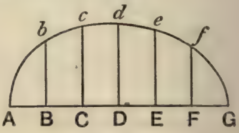


FIG. 56.

EXAMPLES. CIII.

1. Find the areas of the farms represented by the following plans. The numbers indicate chains:

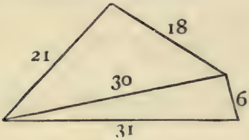


FIG. 57.

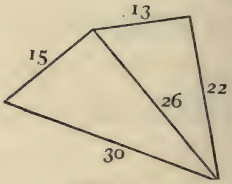


FIG. 58.

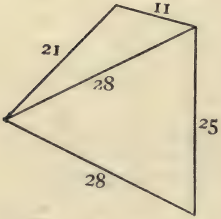


FIG. 59.

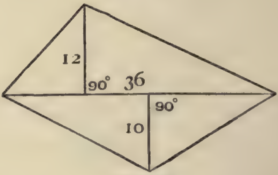


FIG. 60.

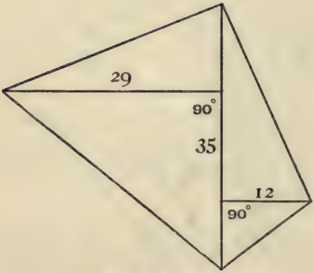


FIG. 61.

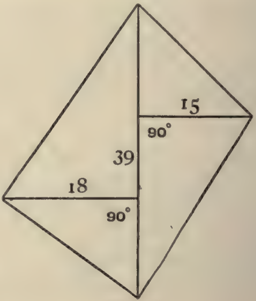
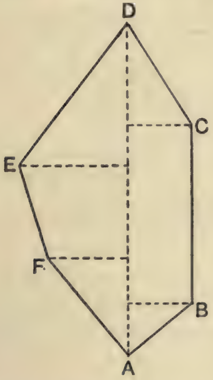


FIG. 62.

2. Find the area of the figure of which the ordinates are 3, 8, 15, 24, 35, 48, 63 yards, and the common distance between the ordinates 2 yards.

Polygons.

The term polygon is applied to plane surfaces bounded by more than four straight sides. Such surfaces, however many sides they may have, can always be divided into triangles and quadrilaterals, the sum of the areas of which is, in each case, equal to the area of the polygon. The following figure, with its accompanying table, is such as a surveyor would make in his survey of a farm of the shape given. The measurements are entered in his sketch book as they are made.



	to D	
	58	
	40	11 to C
19 to E	33	
14 to F	17	
	9	11 to B
From	A	go North

FIG. 63.

The dotted line *AD* is called the base, and all the measurements in the second column are measurements along this line, each being from *A*. Of course, the base line of a survey does not necessarily run North. The offsets are perpendicular to the base. The measurements are in chains. From the foregoing table the area can be computed at the surveyor's leisure. The figure is seen to consist of a rectangle, a trapezium, and four triangles.

Area of the rectangle $= (31 \times 11)$ sq. ch. $= 341$ sq. ch.

Area of triangles East of base $= \left(\frac{11 \times 9}{2} + \frac{18 \times 11}{2}\right)$ sq. ch. $= 148{\cdot}5$ sq. ch.

Area of trapezium $= \frac{33 \times 16}{2}$ $= 264$ sq. ch.

Area of triangles West of base $= \left(\frac{19 \times 25}{2} + \frac{17 \times 14}{2}\right)$ sq. ch. $= 356{\cdot}5$ sq. ch.

$\therefore$ area of $ABCDEF = 1110$ sq. ch.

$= 111$ acres.

EXAMPLES. CIV.

From the following tables draw plans of the fields and compute their areas:

1.

CHAINS.

	To *D*	
	50	
	39	18 to *C*
10 to *E*	28	
	18	20 to *B*
From	*A*	go North.

2.

LINKS.

	To *D*	
	1700	
	1500	400 to *C*
	900	800 to *B*
600 to *E*	600	
From	*A*	go North.

3.

LINKS.

	To *G*	
	2560	
960 to *F*	1880	
	1440	160 to *E*
700 to *D*	1200	
250 to *C*	850	
	600	912 to *B*
From	*A*	

4.

LINKS.

	To *B*	
	1200	160 to *H*
280 to *G*	1120	
300 to *F*	960	
	940	400 to *E*
760 to *D*	760	
	200	30 to *C*
From	*A*	

The Circle.

A circle is a plane figure bounded by a curved line called the *circumference* or *perimeter*, and is such that all straight lines drawn to the circumference from a certain point within it, called the *centre*, are equal to one another.

A *diameter* of a circle is a straight line drawn through the centre and terminated both ways by the circumference.

A *radius* of a circle is a straight line drawn from the centre to the circumference.

Draw several circles of different diameters, and prove by measuring that the following relations are true:

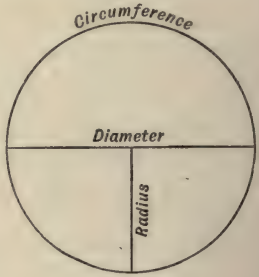


FIG. 64.

(1) *All the diameters of a circle are equal.*

(2) *The radius of a circle is equal to half the diameter.*

(3) *All radii of a circle are equal.*

(4) *The length of the circumference of a circle is very nearly equal to $3\frac{1}{7}$ times the length of the diameter; that is,*

$$\frac{\textit{circumference}}{\textit{diameter}}=\frac{3\frac{1}{7}}{1}.$$

In mathematics, the ratio of the circumference of a circle to the diameter is denoted by the Greek letter π (*pi*). Hence we have

$$\frac{\textit{circumference}}{\textit{diameter}}=\pi, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\textit{circumference}=\pi\times\textit{diameter}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\frac{\textit{circumference}}{\pi}=\textit{diameter}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

To five decimal places the value of π is 3·14159, to four places it is 3·1416, and to three places it is 3·142, the last being the value most suitable for computation by four-figure logarithms.

Ex. 1. *The diameter of a circle is* 3·1 *cm.: find the circumference.*

Denote the diameter by d, and the circumference by c.

Using the ratio $\pi=3{\cdot}14159$, we have

$$\begin{aligned} c&=\pi\times d\\ &=(3{\cdot}14159\times 3{\cdot}1)\text{ cm.}\\ &=9{\cdot}738929\text{ cm.}\end{aligned}$$

Using the ratio $\pi=3\frac{1}{7}$, we have

$$\begin{aligned} c&=\pi\times d\\ &=\left(3\tfrac{1}{7}\times 3{\cdot}1\right)\text{ cm.}\\ &=9{\cdot}74285\text{ cm.}\end{aligned}$$

Ex. 2. *The circumference of a circle is* 154 *yards: find the diameter.*

Using the ratio $\pi=3{\cdot}14159$, we have

$$\begin{aligned} d&=\frac{c}{\pi}\\ &=\frac{154}{3{\cdot}14159}\\ &=49{\cdot}02\text{ yds. (correct to 2 places).}\end{aligned}$$

Using the ratio $\pi=3\frac{1}{7}$, we have

$$\begin{aligned} d&=\frac{c}{\pi}\\ &=\frac{154}{3\frac{1}{7}}\\ &=49\text{ yards.}\end{aligned}$$

Ex. 3. *The cost of fencing a circular piece of ground at* 2*s.* 1*d. per yard was* £59. 11*s.* 8*d.: find the length of a straight path running from side to side through the centre.* (Take $\pi=3\frac{1}{7}$).

Total cost = £59. 11*s.* 8*d.* = 14300*d.*;

cost per yard = 2*s.* 1*d.* = 25*d.*;

$$\therefore\ \text{length of circumference}=\frac{14300}{25}\text{ yds.}=572\text{ yards.}$$

$$\begin{aligned}\text{Diameter}&=\frac{\text{circumference}}{3\frac{1}{7}}\\ &=\frac{572}{3\frac{1}{7}}=182\text{ yds.}\\ &=\text{length of the path.}\end{aligned}$$

The area of a circle is found by multiplying the square of the radius by π; that is,

$$\text{area} = \pi r^2; \quad \dots\dots(1)$$

hence
$$\frac{\text{area}}{\pi} = r^2, \quad \dots\dots(2)$$

$$\frac{\text{area}}{r^2} = \pi, \quad \dots\dots(3)$$

$$\sqrt{\frac{\text{area}}{\pi}} = r. \quad \dots\dots(4)$$

Hence, given any two of these quantities, we can easily find the third.

Concentric Circles.

Circles having the same centre but diameters of different lengths are called *concentric circles.*

Let it be required to find the area of the *plane ring* enclosed between the two circumferences of the accompanying figure.

Denote the radius of the larger circle by R, and that of the smaller by r; then

area of larger circle $= \pi R^2$

and area of smaller circle $= \pi r^2$;

$\therefore$ difference of their areas

$$= \pi R^2 - \pi r^2$$
$$= \pi(R^2 - r^2)$$
$$= \pi(R + r)(R - r)$$

= area of the plane ring enclosed between the circumferences of the circles.

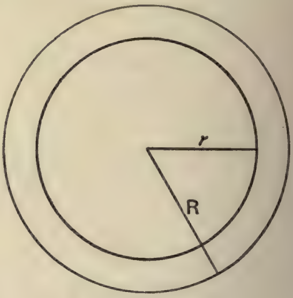


FIG. 65.

Let $r = 9$ and $R = 12$ units of length; then width of ring $= 3$ units. Take $\pi = 3\frac{1}{7}$; then

$$\text{area of ring} = 3\tfrac{1}{7}(12 + 9)(12 - 9)$$
$$= (\tfrac{22}{7} \times 21 \times 3) \text{ sq. units}$$
$$= 198 \text{ sq. units.}$$

EXAMPLES. CV.

(Take $\pi = 3\frac{1}{7}$.)

1. The diameter of a circular lawn is 157½ yds.: find the length of the fence that encloses the lawn.

2. The diameter of a circular garden is 4 chains 72½ links: find the length of the fence that surrounds the garden.

3. The circumference of a circular garden is 12 chains 10 links: find the length of a straight path that runs from side to side through the centre of the garden.

4. The circumference of a circular field is 36 chains 30 links: find the length of a straight road running from side to side through the centre of the field.

5. The circumference of a circular park is 363 poles: find the length of a straight path that runs from the centre to the side of the park.

6. The radius of a circle is 21 yds.: find the area.

7. The diameter of a circle is 63 yds.: find the area.

8. The diameter of a circular field is 7 chains 56 links: find the area.

9. The diameter of a circular garden is 126 yds.: find the area in acres, roods, etc.

10. The circumference of a circle is 58 yds. 2 ft.: find the area in square feet.

11. The circumference of a circle is 279·997 yds.: find the area in square yards.

12. The area of a circular field is 65 ac. 2 ro. 1⅝ po.: find the diameter and the circumference.

13. The area of a circular grass plot is 129 sq. yds. 3 ft. 90 ins.: find the diameter and the circumference.

14. A circular grass plot, the diameter of which is 40 yards, contains a gravel walk, 1 yard wide, running round it 1 yard from the edge: find the cost of asphalting the gravel walk at 4*d.* per square yard.

15. How many trees, at distances of 7½ yds., can be planted round a circular field the radius of which is 105 yds.

16. In a circular riding school of 100 ft. diameter, a circular ride, immediately within the outer edge, is to be made of a uniform width of 10 ft.: find the cost of doing this at 4*d.* per sq. ft.

17. A horse going round a circus ring 20 ft. in diameter was observed to make 13 complete circuits in a minute: express in miles per hour the rate at which he was moving.

18. If a wheel revolves twice in 16½ ft., and if in passing round a circular bowling-green it makes 200 revolutions, what is the area of the green in acres, etc.?

Regular Polygons.

A *regular polygon* is a rectilineal figure having any number of equal sides, and having all its angles equal and all its sides equally distant from the centre.

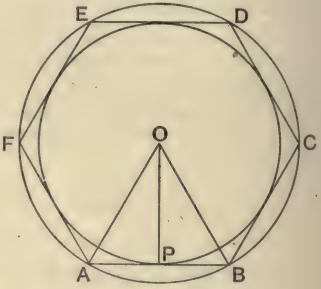


Fig. 66.

$ABCDEF$ is a regular polygon with a circle *circumscribed* about it and a circle *inscribed* in it. Let O be the centre, and let it be joined to the angular points A and B; then, because AO, OB are radii of the circumscribing circle, they are equal. It is obvious that a triangle equal in every respect to the triangle AOB can be erected on each of the other sides of the polygon; hence

$$\text{area of the polygon } ABCDEF = \triangle AOB \times 6$$
$$= \tfrac{1}{2}(AB \times PO) \times 6.$$

Let the polygon have n equal sides, and let the centre O be joined to the extremities of AB; then

$$\text{area of the polygon} = \triangle AOB \times n$$
$$= \tfrac{1}{2}(AB \times \text{height of triangle } AOB) \times n$$
$$= \frac{n}{2}(\text{side} \times \text{perpendicular}).$$

It should be noted that the perpendicular is equal to the radius of the inscribed circle. It is sometimes called the *apothegm* of the polygon.

Polygons are named according to the number of sides they have.

A polygon having five sides is called a pentagon.
„ six sides „ a hexagon.
„ seven sides „ a heptagon.
„ eight sides „ an octagon.
„ nine sides „ a nonagon.
„ ten sides „ a decagon.

The triangles into which a regular hexagon is divided by the straight lines joining the centre to the angular points are equilateral; hence

$$\text{area of the hexagon} = \left(\frac{\text{side}^2 \times \sqrt{3}}{4} \times 6\right) \text{units of area.}$$

EXAMPLES. CVI.

1. The side of a regular pentagon is 36 inches and the perpendicular from the centre 25 inches: find in square ft. the area of the polygon.

2. The side of a garden pentagonal in shape is 2·16 chains and the shortest line from the centre to a side is 1·5 chains: find in sq. chains the area of the garden.

3. Each side of a hexagonal park is 30 chains and the distance from the centre of the park to one of its sides is 25·96 chains: find in acres the area of the park.

4. The side of a regular heptagon is 2·8 cm. and the perpendicular from the centre to the side is 1·9 cm.: find the area of the polygon.

5. A park is in shape a regular octagon, the side of which is 23·75 chains, and the perpendicular from the centre to the side 28·25 chains: find in acres the area of the park.

Arcs, Chords, and Sectors of Circles.

An *arc* of a circle is a part of the circumference, and the straight line joining its extremities is called the *chord* of the arc. Thus, in the figure, ACB and AEB are *arcs*, and the straight line AB, which joins their extremities, is the *chord* of the arcs.

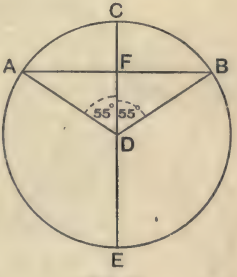


FIG. 67.

The parts into which the circle is divided by AB are called *segments* of the circle.

A *sector* of a circle is the part of the circle bounded by an arc and two radii drawn to the extremities of the arc. Thus, in the figure, ADC, BDC, and AEB are *sectors* of the circle.

The *angle of a sector* is the angle included by the two radii. Thus the angle ADB is the angle of the sector ADB.

The angle ADC is called the *central angle* of the arc AC and the angle ADB the *central angle* of the arc ACB. *In any circle arcs are proportional to their central angles*; hence

$$\frac{\text{arc } AC}{\text{arc } ACB}=\frac{\text{angle } ADC}{\text{angle } ADB}=\frac{55^\circ}{110^\circ}=\tfrac{1}{2}.$$

The sum of all the angles at the point called the centre of a circle is four right angles or 360°, and the circumference is said to subtend an angle of 360°; hence

$$\frac{\text{arc } AC}{\text{circumference}}=\frac{\text{angle } ADC}{360^\circ}=\frac{55}{360}=\frac{11}{72} \text{ of the circumference.}$$

The areas of sectors of the same circle are proportional to the angles of the sectors; hence

$$\frac{\text{sector } ADC}{\text{sector } ADB}=\frac{\text{angle } ADC}{\text{angle } ADB}=\tfrac{55}{110}=\tfrac{1}{2};$$

that is, area of sector $ADC=\frac{1}{2}$ area of sector ADB.

Now, just as the circumference may be regarded as an arc the central angle of which is 360°, so may the circle be regarded as a sector the central angle of which is 360°; hence

$$\frac{\text{area of sector } ADC}{\text{area of the circle}}=\frac{\text{angle } ADC}{360}=\tfrac{55}{360}=\tfrac{11}{72};$$

that is, area of sector $ADB=\frac{11}{72}$ of the area of the circle.

And generally, if the angle of a sector is x°, then

$$\text{area of sector}=\frac{x}{360} \text{ of area of circle}=\frac{x}{360}\pi r^2, \quad \ldots(1)$$

and length of arc subtending the angle $=\frac{x}{360}$ of circumference

$$=\frac{x}{360}\times 2\pi r. \quad \ldots\ldots\ldots\ldots(2)$$

We have seen that

$$\frac{\text{arc of sector}}{\text{circumference}}=\frac{\text{angle subtending the arc of sector}}{360^\circ}$$

and

$$\frac{\text{area of sector}}{\text{area of circle}}=\frac{\text{angle subtending the arc of sector}}{360^\circ};$$

$$\therefore \frac{\text{area of sector}}{\text{area of circle}}=\frac{\text{arc of sector}}{\text{circumference}};$$

$$\text{that is,}\quad \frac{\text{area of sector}}{\pi r^2} = \frac{\text{arc}}{2\pi r}\ (\text{for diameter} = 2r);$$

$$\therefore\ \text{area of sector} = \frac{\text{arc} \times \pi r^2}{2\pi r} = \frac{\text{arc} \times r}{2}$$

$$= \tfrac{1}{2}\ \text{arc} \times \text{radius.} \qquad \text{.......................(3)}$$

The area of the segment ACB = area of sector $ACBD - \triangle ADB$

$$= \tfrac{1}{2}\ \text{arc}\ ACB \times AD - \triangle ADB.$$

Suppose $AD = 35$ and $AB = 42$ feet; then

$$\text{circumference} = (70 \times \tfrac{22}{7})\ \text{ft.} = 220\ \text{ft.}$$

$$\text{Angle } ADB = 110^\circ;$$

$$\therefore\ \text{arc } ACB = (\tfrac{110}{360} \text{ of } 220)\ \text{ft.} = \tfrac{605}{9}\ \text{ft.};$$

$$\therefore\ \text{area of sector} = (\tfrac{1}{2} \times \tfrac{605}{9} \times 35)\ \text{sq. ft.}$$

$$= 1176\tfrac{7}{18}\ \text{sq. ft.}$$

$$\text{Area of triangle } ADB = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{56 \times 21 \times 21 \times 14}$$

$$= 588\ \text{sq. ft.};$$

$$\therefore\ \text{area of segment } ACB = (1176\tfrac{7}{18} - 588)\ \text{sq. ft.}$$

$$= 588\tfrac{7}{18}\ \text{sq. ft.}$$

EXAMPLES. CVII.

$(\pi = \tfrac{22}{7}.)$

1. The arc of a sector of a circle is 19 yards 1 foot 8 inches and the radius of the circle 18 yards 2 feet: find the area of the sector.

2. The arc of a sector of a circle is 55 feet and the radius of it is $26\tfrac{1}{4}$ feet: find the area of the sector.

3. The minute hand of a clock is 10 inches long: find the area on the clock face which it describes between 9 a.m. and 9.35 a.m.

4. The arc of a sector of a circle subtends an angle of 72° and the diameter of the circle is 49 feet: find the area of the sector.

5. The angle subtending a sector of a circle is 72° and the diameter of the circle is 49 feet: find the length of the arc.

6. The arc of a sector is $30\tfrac{4}{5}$ feet and the circumference is 154 feet: find the angle subtending the arc.

7. A field of the shape of an equilateral triangle contains $\tfrac{1}{2}$ an acre: what length of cord is necessary to tether a horse at one corner so as to allow him to feed over half the field?

8. An arc of a circle is 27 feet long and it subtends an angle of 40° at the centre: find the circumference of the circle.

Similar Figures.

Similar figures are figures of the same shape, but not necessarily of the same size. Circles are similar figures; sectors of circles the radii of which include equal angles are similar figures; rectilineal figures having corresponding equal angles and proportional sides are similar figures. *Equality of angles* and *proportionality of sides* are the essential properties of similar rectilineal figures.

In triangles, if either of these properties is present, the other is also present. Hence, if two triangles are equiangular, the sides about their equal angles are proportionals.

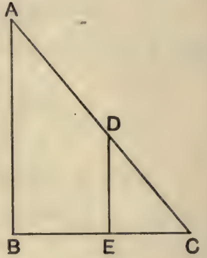


FIG. 68.

Let ABC and DEC be right-angled triangles having the angles at E and B right angles; then

$$\frac{\text{side } EC}{\text{side } ED}=\frac{\text{side } BC}{\text{side } BA}.$$

Hence, if any three of these quantities are given, it is easy to find the fourth. For example, let AB be a telegraph pole and BC its shadow 18 feet long, and let DE be an upright pole 10 feet long and EC its shadow 8 feet long; then

$$\frac{AB}{BC}=\frac{DE}{EC};\quad\text{that is,}\quad\frac{AB}{18}=\frac{10}{8};$$

$$\therefore\ AB=\frac{10\times 18}{8}\text{ ft.}=22\tfrac{1}{2}\text{ ft.},$$

a result the truth of which can be tested on squared paper.

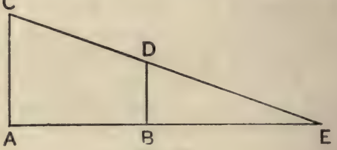


FIG. 69.

Let BE represent an unfordable stream, the width of which a man at B wishes to ascertain. He takes two rods, one 3 feet long and the other 5 feet, fixes the shorter vertically at B, the margin of the stream, and moves back at right angles to the bank of the stream until, resting the other rod vertically on the ground, he sees the upper ends of the rods in line with the further margin of the stream. AC represents the longer and BD the shorter rod in position, and CE the line of sight. The man now measures the distance from A to B, and finds it 42 feet. He now has the data from which to find the width of the stream.

Let x denote the width of the stream in feet; then

$$\frac{x}{BD}=\frac{x+AB}{AC}; \text{ that is, } \frac{x}{3}=\frac{x+42}{5};$$

$$\therefore\ 5x=3x+126,$$

$$\text{whence}\quad x=\tfrac{126}{2}=63 \text{ ft.}$$

Areal Relation of Similar Figures.

The following algebraic proofs of the areal relation of similar figures can be illustrated on squared paper.

1. *Similar triangles*; take the equilateral triangle.

$$\text{Area of equilateral triangle with side } a=\frac{a^2\sqrt{3}}{4},$$

$$\text{area of equilateral triangle with side } b=\frac{b^2\sqrt{3}}{4};$$

$$\therefore\ \frac{\text{area of first triangle}}{\text{area of second triangle}}=\frac{\frac{a^2\sqrt{3}}{4}}{\frac{b^2\sqrt{3}}{4}}=\frac{a^2}{b^2}.$$

2. *Circles.* Let R denote the radius of one circle and r the radius of another; then

$$\text{area of first circle}\quad=\pi R^2,$$

$$\text{area of second circle}=\pi r^2;$$

$$\therefore\ \frac{\text{area of first circle}}{\text{area of second circle}}=\frac{\pi R^2}{\pi r^2}=\frac{R^2}{r^2}.$$

3. *Rectangles.* Let the length be, say, $\frac{5}{3}$ of the breadth, and let B denote the breadth of one and b the breadth of another; then

$$\text{area of first rectangle}\quad=\tfrac{5}{3}B^2,$$

$$\text{area of second rectangle}=\tfrac{5}{3}b^2;$$

$$\therefore\ \frac{\text{area of first rectangle}}{\text{area of second rectangle}}=\frac{\frac{5}{3}B^2}{\frac{5}{3}b^2}=\frac{B^2}{b^2}.$$

And generally: *The areas of similar figures vary as the squares of their corresponding sides.*

From this it follows that,

(1) *if the ratio of the corresponding sides of similar figures is given, the ratio of the areas is found by squaring that ratio ;*

(2) *if the ratio of the areas of two similar figures be expressed as a fraction, the ratio of the corresponding sides is found by extracting the square root of the fraction.*

Ex. 1. *Compare the area of the floor of a circular room* 14 *ft. in diameter with that of one* 7 *ft. in diameter.*

Areas of circles vary as the squares of their diameters ;

$$\therefore\ \frac{\text{area of larger circle}}{\text{area of smaller circle}}=\frac{14^2}{7^2}=\frac{14\times 14}{7\times 7}=\frac{4}{1};$$

$$\therefore\ \text{area of larger circle}=\text{area of smaller circle}\times 4\,;$$

$$\therefore\ \text{area of larger circle}=4\text{ times area of smaller circle.}$$

Ex. 2. *The inner diameter of a circular water pipe is* $3\frac{1}{2}$ *inches: find what diameter a pipe must have to carry* 9 *times as much water at the same pressure.*

Since the pipe is to carry 9 times as much water, a cross section of it must have 9 times the area of a similar section of the smaller pipe.

Corresponding dimensions vary as the square root of the areas ;

$$\therefore\ \frac{\text{diameter}}{3\frac{1}{2}}=\frac{\sqrt{9}}{\sqrt{1}};$$

$$\therefore\ \text{diameter}=3\tfrac{1}{2}\times 3=10\tfrac{1}{2}\text{ inches.}$$

Ex. 3. *A field of* 12 *acres is laid down on a plan to a scale of* 1 *inch to* 20 *ft.: find how much paper it will cover.*

The figures are similar :

$$\therefore\ \frac{\text{area of paper}}{\text{area of land}}=\frac{\text{dimensions on paper}}{\text{corresponding dimensions of the land}};$$

$$\therefore\ \frac{\text{area of paper}}{4840\times 9\times 144\times 12}=\frac{1^2}{(20\times 12)^2};$$

$$\therefore\ \text{area of paper}=\frac{4840\times 9\times 144\times 12}{20\times 20\times 12\times 12}\text{ sq. in.}$$

$$=1306{\cdot}8\text{ sq. in.}$$

EXAMPLES. CVIII.

1. There are two circular parks, one of which is 5 chains in diameter and the other 30 chains in diameter : how many of the smaller are equal to the larger ?

2. A garden is 20 chains long and 16 chains wide : find the dimensions of a similar garden 4 times as large.

3. The area of a circular park, the diameter of which is 20 chains, is $314\frac{2}{7}$ sq. chains: find the diameter of a park the area of which is $78\frac{4}{7}$ sq. chains.

4. If a horse tied to a post by a rope 1·78 chains long can graze upon 1 acre, what length of rope would he need to enable him to graze upon $2\frac{1}{4}$ acres?

5. I have a rectangular field 80 chains long and 60 chains wide, and another similar field containing $213\frac{1}{3}$ acres: find the dimensions of the latter.

6. If a pipe $1\frac{1}{2}$ inches in diameter pours into a cistern 54 gallons in a given time, how much will a pipe 2 inches in diameter pour into it in the same time?

7. If a pipe, the diameter of which is 1·5 inches, fill a cistern in 10 hours, in what time will a pipe, the diameter of which is 3 inches, fill it?

8. A circular pipe, the diameter of which is $1\frac{1}{2}$ inches, fills a tank in 5 hours: find the diameter of the pipe that would fill it in $\frac{45}{49}$ of an hour.

9. A field containing $22\frac{1}{2}$ acres is laid down on a plan to a scale of 1 inch to 10 ft.: find the number of square inches of the plan it will occupy.

10. Determine the scale used in the construction of a plan upon which 14·4 square inches represent an area of 1 acre.

11. The areas of two circles are as 15 to 38: if the diameter of the less is represented by unity, what is the nearest integer expressing the circumference of the greater in terms of this unit?

12. The shadow cast by a telegraph pole is 20 feet long and that cast at the same time and place by a vertical pole 6 feet long is 4 feet: find the height of the telegraph pole.

13. Wishing to know the height of a tower, I fix a pole 11 feet long at a distance of 30 feet from the base of the tower; retiring 4 feet farther from the tower, I see the top of the pole in line with the top of the tower: given that my eye is 5 feet above the ground, find the height of the tower. (Prove your answer by using squared paper.)

14. Describe how you could find the width of a river without crossing it.

CHAPTER XXXV.

MENSURATION OF SOLIDS.

Fig. 70 represents a cuboid or rectangular block, that is, a solid bounded by three pairs of rectangular faces, the length, breadth and height of which are not all equal. A brick will serve to illustrate this type of solid.

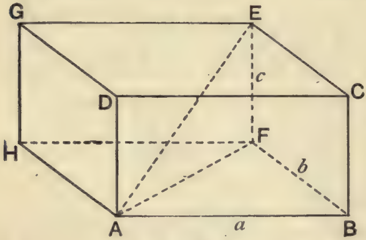


Fig. 70.

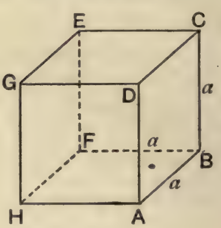


Fig. 71.

Fig. 71 represents a cube, that is, a solid bounded by six equal square faces.

Properties of rectangular solids:

Rectangular solids have six rectangular faces, the opposite ones being equal in every respect.

Their entire surface is the sum of the surfaces of their faces.

In the cuboid there are three pairs of equal faces; in the cube there are six equal faces.

The surface of Fig. $70 = 2ABCD + 2BFEC + 2CEGD$

and the surface of Fig. $71 = 6ABCD$.

Hence, denoting AB by a units of length, BF by b units, and EF by c units, we have

$$\text{surface of Fig. 70} = 2ab + 2bc + 2ac$$
$$= 2(ab + bc + ac) \text{ units of area;}$$

and $\quad$ surface of Fig. $71 = 6a^2$ units of area.

In Fig. 70, $ABFH$ is a rectangle;

$$\therefore\ (AF)^2 = (AB)^2 + (BF)^2;$$

and because the angle AFE is a right angle,

$$(AE)^2 = (AF)^2 + (EF)^2$$
$$= (AB)^2 + (BF)^2 + (EF)^2$$
$$= a^2 + b^2 + c^2;$$

$$\therefore\ \text{diagonal } AE = \sqrt{a^2 + b^2 + c^2}.$$

In the cube (Fig. 71) $a = b = c$;

$$\therefore\ \text{diagonal of cube} = \sqrt{a^2 + a^2 + a^2}$$
$$= \sqrt{3a^2}$$
$$= a\sqrt{3}$$
$$= \text{side} \times \sqrt{3}.$$

The volume or content of a rectangular body a units long, b units broad, and c units high (or deep) is $(a \times b \times c)$ cubic units, that is, abc cubic units.

Hence the volume of a rectangular body

$$= \text{length} \times \text{breadth} \times \text{height}, \quad \text{.................}(1)$$

from which it follows that

$$\text{height} = \frac{\text{volume}}{\text{length} \times \text{breadth}}, \quad \text{..........................}(2)$$

$$\text{breadth} = \frac{\text{volume}}{\text{length} \times \text{height}}, \quad \text{..........................}(3)$$

$$\text{length} = \frac{\text{volume}}{\text{breadth} \times \text{height}}. \quad \text{..........................}(4)$$

The volume of a cube the length of the side of which is a units is $(a \times a \times a)$ cubic units, that is, a^3 cubic units; hence

its side or edge $= \sqrt[3]{a^3} = a$ linear units.

To find the weight of a given volume of matter, it is necessary to know the *relative density* or *specific gravity* of the matter, that is, the number that expresses the ratio of its weight to the weight of an equal volume of water. A cubic foot of water weighs 62·3 lbs.; hence a cubic foot of a substance the specific gravity of which is 2 would be $(62{\cdot}3 \times 2)$ lbs.

Ex. 1. *A rectangular solid is* 2·7 *feet long,* 1·9 *feet wide, and* 1·4 *feet high: find its weight, the specific gravity of the material being* 3·7.

$$\text{Volume} = (2{\cdot}7 \times 1{\cdot}9 \times 1{\cdot}4) \text{ cubic feet;}$$
$$\therefore \text{ weight} = (2{\cdot}7 \times 1{\cdot}9 \times 1{\cdot}4 \times 62{\cdot}3 \times 3{\cdot}7) \text{ lbs.}$$
$$= 1656 \text{ lbs. (by logarithms)}$$
$$= 14 \text{ cwt. } 88 \text{ lbs.}$$

When the specific gravity of a substance and the weight of it are known, it is easy to find the volume; for

$$\text{volume in cubic feet} \times 62{\cdot}3 \times \text{specific gravity} = \text{weight in lbs.;}$$
$$\therefore \text{ volume} = \frac{\text{weight in lbs.}}{62{\cdot}3 \times \text{specific gravity}}.$$

Ex. 2. *The weight of a cubical block of rock of specific gravity* 3·7 *is* 1656 *lbs.: find the volume and the length of the block.*

$$\text{Volume} = \frac{1656}{62{\cdot}3 \times 3{\cdot}7} = 7{\cdot}183 \text{ cub. ft. (by logarithms);}$$
$$\therefore \text{ length of edge} = \sqrt[3]{7{\cdot}183} = 1{\cdot}93 \text{ ft. (by logarithms).}$$

Prism.

The term *prism* or *right prism* is a general name for any solid the ends of which are similar, equal, and parallel plane surfaces, and the edges of which are parallel; it therefore includes the cuboid, the cube, and even the right cylinder whose ends may be regarded as polygons of an infinite number of sides.

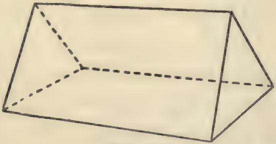


FIG. 72.

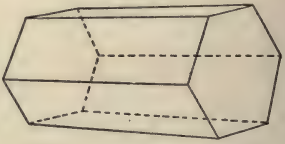


FIG. 73.

The faces of the prisms are seen to be rectangles, and the ends to be triangles in the one case and hexagons in the other.

Obviously, area of the surface of either prism

= sum of areas of side faces + sum of areas of ends

and volume = area of end × length (or altitude).

Cylinder.

AB represents a cylinder, and *ABCD* a piece of paper that has been wrapped round the cylinder so as to cover it exactly, and then laid out on a plane surface. The shape of the paper is seen to be a rectangle the height of which is the height of the cylinder, and the length the length of the circumference of the cylinder.

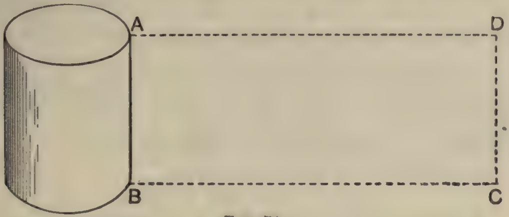


FIG. 74.

The length of the circumference $= \pi d = 2\pi r$.

Denote the height by h; then

$$\text{area of curved surface of cylinder} = 2\pi r h \text{ units.}$$

$$\text{Area of one end} = \pi r^2 \text{ units;}$$

$$\therefore \text{ area of the two ends} = 2\pi r^2 \text{ units;}$$

$$\therefore \text{ area of the surface of the entire cylinder} = (2\pi r h + 2\pi r^2) \text{ units}$$
$$= 2\pi r(h+r) \text{ units.}$$

$$\text{Volume of cylinder} = \text{area of end} \times h$$
$$= \pi r^2 h \text{ cubic units.}$$

The surface and volume of a hollow cylinder open at the ends like a pipe.

Denote the height by h, the external radius by r, and the thickness of the shell of the cylinder by d; then $(r-d) =$ the internal radius.

$$\text{External curved surface} = 2\pi r h$$

$$\text{and internal curved surface} = 2\pi(r-d)h.$$

$$\text{Area of each ring forming the ends} = \pi r^2 - \pi(r-d)^2$$
$$= \pi\{r^2 - (r-d)^2\};$$

$$\therefore \text{ area of the two rings} = 2\pi\{r^2 - (r-d)^2\}.$$

To obtain surface of the hollow cylinder, we add these three results together.

The volume of material in the shell of the cylinder is the difference between the volume of a solid cylinder having the given external dimensions and the volume of a solid cylinder having the internal dimensions; hence

$$\begin{aligned}\text{volume of shell} &= \pi r^2h - \pi(r-d)^2h \\ &= \pi h\{r^2-(r-d)^2\}.\end{aligned}$$

We shall now apply these results to the solution of a question.

Ex. 3. *A cast-iron pipe, open at both ends, is* 14 *feet long, and its external and internal diameters are* 7 *inches and* 6 *inches respectively: find its entire surface and its volume.*

External radius $= 3\frac{1}{2}$ inches; internal radius $= 3$ inches;

$$\therefore \text{ thickness of the iron} = \tfrac{1}{2} \text{ inch.}$$

$$\text{Length of pipe} = 168 \text{ inches.}$$

$$\begin{aligned}\text{External curved surface} &= 2\pi \times \tfrac{7}{2} \times 168 = 1176\pi \text{ sq. in.,} \\ \text{internal curved surface} &= 2\pi \times 3 \times 168 = 1008\pi \text{ sq. in.,} \\ \text{surface of end rings} &= 2\pi\{(\tfrac{7}{2})^2 - (\tfrac{7}{2}-\tfrac{1}{2})^2\} \\ &= 2\pi(\tfrac{49}{4}-9) = 2\pi \times \tfrac{13}{4} = 6\tfrac{1}{2}\pi \text{ sq. in.;} \\ \therefore \text{ whole surface} &= \pi(1176+1008+6\tfrac{1}{2}) \text{ sq. in.} \\ &= (2190\tfrac{1}{2} \times \tfrac{22}{7}) \text{ sq in.} \\ &= 6884\tfrac{3}{7} \text{ sq. inches.} \\ \text{Volume of pipe} &= \pi h\{(\tfrac{7}{2})^2 - (\tfrac{7}{2}-\tfrac{1}{2})^2\} \text{ cubic inches} \\ &= \pi h(\tfrac{49}{4}-9) \text{ cubic inches} \\ &= (\tfrac{22}{7} \times 168 \times \tfrac{13}{4}) \text{ cubic inches} \\ &= 1716 \text{ cubic inches.}\end{aligned}$$

Solid Circular Rings.

The accompanying figure represents a solid circular ring of which *AB* is a cross section. Such a ring may be regarded as a cylinder bent round till both ends meet. Hence the curved surface of the ring is found in the same way as that of a cylinder. It is, however, obvious that neither the inner nor the outer circumference may be taken as the measure of the height or length of the ring, for the one is too short and the other too long; the mean circumference is taken as the

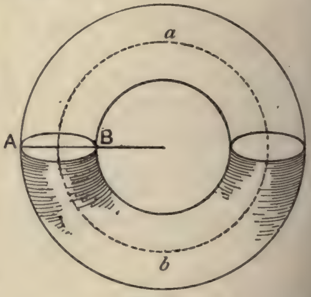


Fig. 75.

measure of the length of the ring. In the figure the mean circumference is represented by the curved line *ab*.

Let R represent the radius of the circle formed by the outer and r the radius of the circle formed by the inner circumference of the ring; then the diameter of the cross section $AB = R - r$, and the radius of the section $= \frac{R-r}{2}$; and the

$$\text{mean circumference} = \frac{\text{outer circumference} + \text{inner circumference}}{2}.$$

$$\text{Volume of ring} = \text{area of cross section} \times \text{mean circumference}$$
$$= \pi\left(\frac{R-r}{2}\right)^2 \times \text{mean circumference};$$

and curved surface of ring

$$= \text{circumference of cross section} \times \text{mean circumference}.$$

EXAMPLES. CIX.

1. A right prism, 25 inches long, stands upon a triangular base the sides of which are 8, 11, and 14 inches: find (1) its surface, (2) its volume, and (3) its weight, the specific gravity of its material being 7·85.

2. A circular plate of iron weighs $4\frac{2}{5}$ ounces per cubic inch; it is $14\frac{1}{8}$ inches in diameter and 2 inches thick: find its weight to the nearest pound.

3. Find to the nearest penny the cost of painting the convex surface of 5 cylindrical pillars, each 14 feet high and a foot in diameter, at $8\frac{1}{2}$*d.* per square yard

4. The mean circumference of a cylindrical ring is 30 inches, and the volume of the ring is $13\frac{1}{4}$ cubic inches: what is its thickness?

5. The diameter of a circular well is 4 feet 8 inches and its depth 30 feet: find, to the nearest penny, the cost of excavating it at 7*s.* 6*d.* per cubic yard.

6. The dimensions of a rectangular cistern are 12, 9, and 6 ft.: what must be the depth of a second cistern equal in content to the first, supposing the bottom of the second to be a square having its side 9 ft.?

7. A room 11 ft. high is half as long again as it is wide, and its cubical content is $4768\frac{1}{2}$ ft.: find its length and breadth.

8. The content of a box, whose length is twice its breadth and whose breadth is twice its depth, is 1 cub. yd.: find its dimensions, and the cost of gilding it at 1*s.* 9*d.* per sq. ft.

9. Find the cost of painting the surface of a cube, the edge of which is 2 ft. 5 ins., at 1*s.* 3*d.* per sq. yd.

10. A rectangular solid 10 ft. long and 8 ft. broad weighs $4\frac{1}{2}$ tons, and a cubic inch of the same material weighs 2 ozs. avoirdupois: what is the depth of the solid?

11. Find the thickness and solid content of an armour-plate, 22 ft. by 17 ft., weighing 34 tons, having given that $5\frac{1}{2}$ cub. ft. of iron weigh one ton.

12. How many cubic yards must be dug out to make a circular well 3 ft. in diameter and 30 ft. deep?

13. A cylindrical well is sunk to the depth of 119 ft. and it measures 4 ft. across at the mouth: find how many cubic feet of earth were taken out of it.

14. Find the number of cubic yards of earth dug out of a tunnel 100 yds. long, whose section is a semi-circle with a radius of 10 ft.

15. The cost of excavating a circular well, of which the depth was 45 ft. and the diameter 3 ft. 9 ins., was £8. 11*s.* $10\frac{1}{2}$*d.*: find the cost per cubic yard.

16. What was the cost of painting the walls of a circular room 16 ft. high and 18 ft. in diameter, at $7\frac{1}{2}$*d.* per sq. yd.?

17. The top of a circular table is 7 ft. in diameter and 1 in. thick: how many cubic feet of wood does it contain, and what would be the cost of polishing the upper surface at 6*d.* per sq. ft.?

18. Two iron bars are of the same weight; one of them is round and $3\frac{1}{2}$ ins. in diameter and 3 ft. long; the other is square and 3 ins. thick: find the length of the latter.

19. Find the cubic content of a triangular prism, the sides of the base being 4 ft., 13 ft., and 15 ft. respectively, and the height of the prism 14 ft.

20. Find the cubic content of a right-angled triangular prism, the side of which is 1 ft. 3 ins., the hypotenuse 2 ft. 1 in., and the height 35 ft.

21. Find the cubic content of a triangular prism, the sides of the base of which are 9 ft., 10 ft., and 17 ft. respectively, and the height of the prism 40 ft.

22. Find the volume of a hexagonal prism, the side of the hexagon being 10 ins. and the height of the prism 20 ft.

23. A wall with slanting sides is 64 ft. long and 6 ft. high with its top and base in parallel planes: if it is 12 ft. wide at the bottom and 8 ft. wide at the top, what is its cubic content?

24. A railway cutting with its top and bottom in approximately parallel planes is 12 ft. wide at the bottom and 26 ft. wide at the top; its depth is 18 ft. and its length 75 yards: find its approximate cubic content.

25. Find the cubic content of a cylindrical shell the inner and outer diameters of which are 33 inches and 37 inches respectively, and the length 12 ft.

26. A cylindrical roller is 44 inches in circumference: how far from the end must I make the section to cut off 3 cub. ft. from the roller?

27. Find the content of a round pillar the length of which is 20 ft. and the circumference 5½ ft.

28. A court-yard is to be paved; its form is that of a trapezium, the diagonal of which is 65 ft., the perpendiculars upon the diagonal from opposite corners being 33·5 ft. and 28 ft. respectively: find how many square yards there are to be paved.

29. How many cubic inches are there in a solid cylindrical ring, the outer and inner diameters of which are 11 and 8 inches respectively?

30. The diagonal of a cubical cistern is 7 ft.: find how many gallons the cistern will hold, supposing a gallon to be 277¼ cubic inches.

31. What is the area of a sector of a circle, the length of the arc being 8·156 inches and the radius 1 ft.?

32. A rectangular court is twice as long as it is broad, and its area is equal to that of a circle the radius of which is 35 ft.: find the sides of the rectangle.

33. By how many square inches does the exterior convex surface of an iron pipe exceed the interior concave surface if the length of the pipe is 9 ft., the bore 3¼ inches, and the thickness of the metal ¾ of an inch?

34. Find the surface of a triangular prism the length of which is 10 inches, and the sides 5, 5, and 6 inches respectively.

35. Find the number of gallons of water in a circular tank the diameter of which is 10 yards and the depth 5 ft., supposing a gallon to contain 277¼ cubic inches.

The Pyramid.

A *pyramid* is a solid bounded by plane faces of which the base is any rectilineal figure, and the rest of the faces triangles having a common vertex not in the plane of the base. The figures given above represent *right* pyramids, the one on a square base and the other on a regular pentagonal base. The entire surface of either

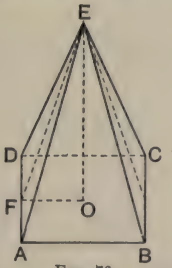


FIG. 76.

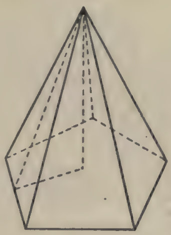


FIG. 77.

of the pyramids is obviously the sum of the areas of the sloping sides plus the area of the base.

Take the pyramid on a square base, and let EF be perpendicular to AD; then

$$\text{area of } ADE = \frac{EF \times AD}{2}$$
$$= \tfrac{1}{2}(\text{slant height} \times \text{base});$$

$\therefore$ area of the four sides of the pyramid $= \tfrac{4}{2}(\text{slant height} \times \text{base})$; and, generally,

area of sides of pyramid of n equal sides $= \frac{n}{2}(\text{slant height} \times \text{base})$.

For the surface of the whole pyramid, add to this result the surface of the base.

If the slant height is not given, it is easily found; for, since the angle at the centre O is a right angle,

$$(EF)^2 = (OF)^2 + (OE)^2;$$
$$\therefore\ EF = \sqrt{(OF)^2 + (OE)^2}.$$

The accompanying figure represents a cube divided into six pyramids. The base of each pyramid is seen to be one of the faces of the cube, and the centre of the cube is seen to be the common vertex of the pyramids. The distance from the centre of the cube to the base of each pyramid is therefore the height of the pyramid. If half the cube were cut away, the volume of the remainder would be equal to the volume of three of the pyramids, and $\frac{1}{3}$ of this volume would be equal to the volume of one of them.

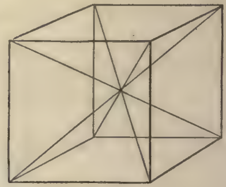


FIG. 78.

That is, volume of pyramid $= \tfrac{1}{3}(\text{area of base} \times \text{height})$,

or expressed in words:

The volume of a pyramid is equal to one-third of the volume of a prism on the same base and of the same altitude.

The Cone.

A cone may be regarded as a pyramid having for its base a polygon of an infinite number of sides; that is, a pyramid becomes a cone when its base is a circle; and therefore the rules for finding the surface and the volume of the cone correspond with those given above for finding the surface and the volume of the pyramid.

Let ACD be a right cone and AB the vertical drawn from the vertex A to the centre B of the base. Denote the slant height AC by l, the height AB by h, and the radius CB by r; then

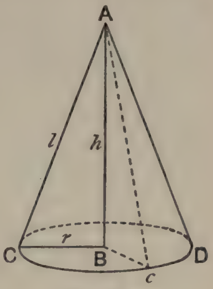


FIG. 79.

$$l^2 = r^2 + h^2; \quad \therefore \ l = \sqrt{r^2 + h^2};$$

and $$h^2 = l^2 - r^2; \quad \therefore \ h = \sqrt{l^2 - r^2}.$$

Hence, if any two of these three dimensions are given, it is easy to find the third. If r is given, the circumference of the base is virtually given, for the circumference $= 2\pi r$.

Curved surface of cone

$= \frac{1}{2}$(sum of infinite number of sides of enclosed polygon × slant height)

$= \frac{1}{2}$(circumference of base of cone × slant height)

$= \frac{1}{2}(2\pi r \times$ slant height)

$= \pi r \times$ slant height.

We may illustrate the truth of these results as follows:

The accompanying figure represents a hollow cone slit down the side from vertex to base, and then laid out on a plane surface, where it is seen to have the shape of a sector of a circle. The side h is the radius of the circle and also the *slant height* of the cone. Hence

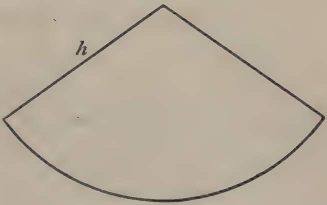


FIG. 80.

curved surface of cone = area of sector

$= \frac{1}{2}(\text{arc} \times r)$

$= \frac{1}{2}$(circumference of base of cone × slant height)

$= \frac{1}{2}(2\pi r \times$ slant height)

$= \pi r \times$ slant height.

The volume of a cone is equal to $\frac{1}{3}$ of the volume of the cylinder on the same base and of the same height; hence

$$\begin{aligned}\text{volume of cone} &= \tfrac{1}{3}(\text{area of base} \times \text{height})\\ &= \tfrac{1}{3}(\pi r^2 \times \text{height}).\end{aligned}$$

The Sphere.

The cubic content or volume of a sphere is two-thirds of that of a cylinder the height and diameter of which are the same as those of the sphere.

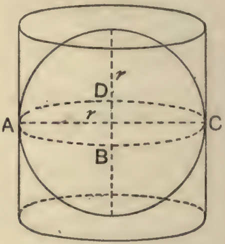


FIG. 81.

The figure is a representation of such a cylinder and sphere. *ABCD* is a plane cutting the sphere into hemispheres. The plane passes through the centre of the sphere, and is therefore called a *great circle* or a *plane central section* of the sphere. Obviously, the area of the plane $= \pi r^2$, where $r =$ radius.

The surface of a sphere is four times the surface of a plane central section of the sphere; hence

$$\text{surface of sphere} = 4\pi r^2. \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The formula for the volume of a sphere is

$$\text{volume of sphere} = \tfrac{4}{3}\pi r^3. \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\text{Since}\quad \text{surface of sphere} = 4\pi r^2,$$

$$\therefore\ r = \sqrt{\frac{\text{surface}}{4\pi}};\ \ldots\ldots\ldots\ldots\ldots(3)$$

$$\text{and, since}\quad \text{volume} = \tfrac{4}{3}\pi r^3,$$

$$\therefore\ r = \sqrt[3]{\frac{\text{volume} \times 3}{4\pi}}\ \ldots\ldots\ldots\ldots\ldots(4)$$

Ex. 1. *Find the surface and the volume of a sphere the radius of which is* 2·7 *inches.* ($\pi = 3\cdot142$.)

$$\text{Surface of sphere} = 4\pi r^2 = 4 \times 3\cdot142 \times (2\cdot7)^2 \text{ sq in.}$$

Let x denote the area; then

$$\begin{aligned}\log x &= \log 4 + \log 3\cdot142 + 2\log 2\cdot7\\ &= \cdot6021 + \cdot4972 + \cdot8628\\ &= 1\cdot9621, \text{ the antilog of which is } 9164;\end{aligned}$$

$\therefore$ area of sphere $= 91\cdot6\ldots$ sq. inches.

$$\text{Volume of sphere} = \tfrac{4}{3}\pi r^3 = \tfrac{4}{3} \times 3{\cdot}142 \times (2{\cdot}7)^3.$$

Let x denote the volume; then

$$\begin{aligned}\log x &= \log \tfrac{4}{3} + \log 3{\cdot}142 + 3 \log 2{\cdot}7\\ &= {\cdot}6021 + {\cdot}4972 + 1{\cdot}2942 - {\cdot}4771\\ &= 1{\cdot}9164, \text{ the antilog of which is } 8249;\end{aligned}$$

$\therefore$ volume of sphere $= 82{\cdot}5$ cubic inches (nearly).

Ex. 2. *What is the area of each triangular face of a square pyramid the area of the base of which is* 36 *square feet and the height* $4\frac{1}{2}$ *feet? Also, what is the diameter of a sphere equal in cubic content to the pyramid?*

$$\text{Side of base} = \sqrt{36} = 6 \text{ feet};$$

$$\therefore \text{ distance from centre of base to side} = 3 \text{ feet.}$$

$$\text{Vertical height of pyramid} = 4\tfrac{1}{2} \text{ feet};$$

$$\therefore \text{ slant height of side} = \sqrt{3^2 + (4\tfrac{1}{2})^2} = \sqrt{\frac{117}{4}} = \frac{\sqrt{117}}{2};$$

$$\begin{aligned}\therefore \text{ area of one face} &= \tfrac{1}{2}\left(\frac{\sqrt{117}}{2} \times 6\right) = \frac{\sqrt{117} \times 3}{2}\\ &= 16{\cdot}22 \text{ sq. ft. (the root extracted by logs).}\end{aligned}$$

$$\begin{aligned}\text{Volume of pyramid} &= \tfrac{1}{3}(\text{area of base} \times \text{height})\\ &= \tfrac{1}{3}(36 \times 4\tfrac{1}{2}) = 54 \text{ cubic feet,}\end{aligned}$$

which is also the volume of the sphere the diameter of which is required.

$$\begin{aligned}\text{Now,}\quad r \text{ (radius of sphere)} &= \sqrt[3]{\frac{\text{volume} \times 3}{4\pi}}\\ &= \sqrt[3]{\frac{54 \times 3}{4 \times \frac{22}{7}}}\\ &= \sqrt[3]{\frac{162}{\frac{88}{7}}};\end{aligned}$$

$$\begin{aligned}\therefore\ \log r &= \tfrac{1}{3} \log 162 - \tfrac{1}{3}(\log 88 - \log 7)\\ &= {\cdot}7365 - {\cdot}3665\\ &= {\cdot}3700, \text{ the antilog of which is } 2344;\end{aligned}$$

$$\therefore\ r = 2{\cdot}344 \text{ feet};$$

whence diameter $= (2{\cdot}344 \times 2)$ feet $= 4{\cdot}69$ feet (nearly).

Oblique Figures.

The figure to the right represents a pack of cards pushed from a rectangular solid into an *oblique* solid. The dotted lines indicate the positions of the ends when the pack formed a rectangular solid. Obviously, the change of form has made no change in the volume of the pack. Hence the volume of an *oblique prism* is equal to the volume of a *right prism* on the same base and of the same height, that is, the height measured at right angles to the base. In the figure the height is indicated by the dotted vertical lines.

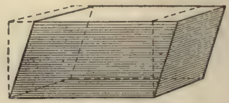


FIG. 82.

What is here said of prisms applies to cylinders, pyramids, and cones.

The volume of any prism = (area of base × height).

The volume of an irregular solid may be found by displacement of water, or by weighing the solid and comparing its weight with that of a known volume of the same material. For example, if it displaces 2·5 cubic inches of water, its volume is 2·5 cubic inches; if a cubic inch of the material weighs 15 grams, and the solid volume of which is to be found weighs 75 grams, the required volume is $\frac{75}{15}$ cubic inches, that is, 5 cubic inches.

EXAMPLES. CX.

1. The circumference of the base of a cone is 12 ft. 6 ins., and its altitude is 8 ft. 3 ins.: find its volume.

2. The diameter of the base of a cone is 14 ins. and the height is 7 ft.: find its volume.

3. The base of a pyramid is a square, the side of which is 3 ft. 4 ins. and the height 27 ft.: find its volume.

4. The diameter of a sphere is 14 ft.: find its volume.

5. The diameter of a sphere is 3 ft. 6 ins.: find its volume.

6. A pyramid stands on a square base, the side of which is 12 ft. and the slant height is 25 ft.: find the whole superficies of the pyramid.

7. What length of canvas $\frac{3}{4}$ yd. wide is required to make a conical tent 12 ft. in diameter at the base and 8 ft. high?

8. What length of canvas $\frac{2}{3}$ yd. wide is required to make a conical tent 16 yds. in diameter at the base and 10 ft. high?

9. A conical wine glass is 2 ins. wide at the top and 3 ins. deep: find how many cubic inches of wine it will hold.

10. How many square inches of gold leaf will gild a globe 18 ins. in diameter?

11. Find the cost of gilding a ball, the radius of which is 18 ins., at 3*s.* 6*d.* per square foot.

12. Find the cost of enamelling the surface of a sphere 5 ins. in diameter, at 5*s.* 2½*d.* per square inch.

13. How much canvas will be required to make a conical tent, the perpendicular height of which is to be 10 ft. and the diameter on the ground 19 ft. 10 ins.?

14. A hollow spherical shell of metal has an internal diameter of 10 ins., and the thickness of the metal is 1 in.: find the number of cubic inches of metal in the shell.

15. How many spherical bullets, each ½ inch in diameter, can be cast from a rectangular block of lead 1 ft. 3 ins. by 1 ft. 2 ins. by 5 ins.?

16. Find in inches the solidity and the surface of a spherical body, the diameter of which is ¾ of a foot.

17. A pyramid having a square base is 40 ft. high: if the side of the base is 3½ ft. and a cubic foot weighs 280 lbs., find the weight of the pyramid.

18. Find the volume of a pyramid, the height of which is 50 ft. and the base a regular hexagon, each side of which is 3 yds.

19. A spherical shell is 7 ins. in external diameter and 2 ins. thick: find what weight of powder will fill it, if 30 cub. ins. of powder weigh 1 lb.

20. A cylinder 5 ft. long and 3 ft. in diameter is closed by a hemisphere at each end: find the whole surface.

The Relation of Volumes of Similar Solids.

Similar solids are solids having the same shape but not necessarily the same size. For example, cubes are similar solids; spheres are similar solids; right prisms having their heights proportional to the corresponding sides of their bases are similar solids.

1. *The Sphere.* Let D denote the diameter of one sphere and d the diameter of another; then

$$\text{volume of first sphere} = \tfrac{4}{3}\pi\left(\frac{D}{2}\right)^3,$$

$$\text{volume of second sphere} = \tfrac{4}{3}\pi\left(\frac{d}{2}\right)^3;$$

$$\therefore\ \frac{\text{volume of first sphere}}{\text{volume of second sphere}} = \frac{\tfrac{4}{3}\pi\left(\frac{D}{2}\right)^3}{\tfrac{4}{3}\pi\left(\frac{d}{2}\right)^3} = \frac{D^3}{d^3}.$$

2. *The Prism.* Let the lengths, breadths, and depths be, say, as 6 : 4 : 3 ; then

$$\begin{aligned} \text{length} &= 2 \text{ depth} \\ \text{and} \quad \text{breadth} &= \tfrac{4}{3} \text{ depth}; \\ \therefore \quad \text{volume} &= 2 \text{ depth} \times \text{depth} \times \tfrac{4}{3} \text{ depth} \\ &= \tfrac{8}{3} (\text{depth})^3. \end{aligned}$$

Let D denote the depth of one such solid and d the depth of another; then

$$\begin{aligned} \text{volume of first} &= \tfrac{8}{3}D^3, \\ \text{volume of second} &= \tfrac{8}{3}d^3; \end{aligned}$$

$$\therefore \frac{\text{volume of first}}{\text{volume of second}} = \frac{\frac{8}{3}D^3}{\frac{8}{3}d^3} = \frac{D^3}{d^3}.$$

In the same way their volumes may be shown to vary as $L^3 : l^3$ or as $B^3 : b^3$, where L, l, B, b, represent lengths and breadths respectively.

And, generally: *The volumes of similar solids vary as the cubes of their corresponding dimensions.*

Hence, if the ratio of the volumes be expressed as a fraction, the ratio of corresponding dimensions is found by extracting the cube root of the fraction; and, conversely, if the ratio of corresponding dimensions be expressed as a fraction, the ratio of the volumes is found by cubing the fraction.

When two similar solids are composed of the same material, or of materials of the same specific gravity, corresponding dimensions vary as the cube root of the weights.

Let L denote the length of the edge of one cube and l the length of another; then

$$\begin{aligned} \text{area of the surface of the first cube} &= 6L^2, \\ \text{area of the surface of the second cube} &= 6l^2; \end{aligned}$$

$$\therefore \frac{\text{area of the surface of the first cube}}{\text{area of the surface of the second cube}} = \frac{6L^2}{6l^2} = \frac{L^2}{l^2}.$$

Let D denote the diameter of one sphere and d the diameter of another; then

$$\begin{aligned} \text{area of the surface of the first sphere} &= 4\pi\left(\frac{D}{2}\right)^2, \\ \text{area of the surface of the second sphere} &= 4\pi\left(\frac{d}{2}\right)^2; \end{aligned}$$

$$\therefore\ \frac{\text{area of the surface of the first sphere}}{\text{area of the surface of the second sphere}}=\frac{4\pi\left(\frac{D}{2}\right)^2}{4\pi\left(\frac{d}{2}\right)^2}=\frac{D^2}{d^2}.$$

And, generally: *The areas of the surfaces of similar figures vary as the squares of their corresponding dimensions.*

Hence, if the ratio of the areas be expressed as a fraction, the ratio of corresponding dimensions is found by extracting the square root of the fraction; and, conversely, if the ratio of corresponding sides be expressed as a fraction, the ratio of the areas is found by squaring the fraction.

Ex. 1. *The edges of two cubes are as* 7 : 5 *: find the ratio of their surfaces and of their volumes.*

$$\text{The ratio of their surfaces}=\frac{7^2}{5^2}=\frac{49}{25};$$

that is, the surface of the first is nearly twice that of the second.

$$\text{The ratio of their volumes}=\frac{7^3}{5^3}=\frac{343}{125};$$

that is, the volume of the larger is nearly $2\frac{3}{4}$ times that of the smaller.

Ex. 2. *The weights of two similar cones of the same material are as* 729 *to* 343 *: find the ratio of the radii of the bases of the cones.*

$$\text{The ratio of the radii}=\frac{\sqrt[3]{729}}{\sqrt[3]{343}}=\frac{9}{7}.$$

EXAMPLES. CXI.

1. The edges of two cubes are as 5 to 4: find the ratio of their surfaces and of their volumes.

2. The diameters of two spheres are as 3 to 7: find the ratio of their surfaces and of their volumes.

3. The weights of two similar spheres are as 2197 and 512: find the ratio of the diameters of the spheres.

4. The diameter of a sphere is 8 inches: find the diameter of a sphere of 27 times the volume of the same material.

5. A sphere 3 inches in diameter weighs 18 lbs.: find the weight of another sphere of the same material that has a diameter of 4 inches.

6. A cubical box 4 feet long holds 64 bushels of corn: find how many bushels a cubical box 6 feet long will hold.

7. If the weight of a leaden spherical ball of 1 inch diameter be $\frac{3}{14}$ lb., find the diameter of a similar ball of the same material that weighs $21\frac{7}{9}$ lbs.

8. Compare the volume of a sphere that touches the six faces of a cube with that of a sphere that passes through the angular points of the cube.

MISCELLANEOUS EXAMPLES. I.

1. If $x=2$ and $y=-\frac{1}{2}$, find the numerical value of $3(x+y)-2(x-y)$.

2. Subtract $\quad 6x+1-2(x+5y)-\{2-(x+2y-1)\}$
from $\quad 2(2-y)-7x-2\{1-3y+6(y-x)\}$.

3. Simplify $a-\{b-(2b+x)\}+\{b-(x-2b)\}$.

4. Multiply $2x+3y$ by $2x-3y$.

5. Multiply $4a^2-6a+9$ by $2a+3$.

6. Divide $30a^3b^2c$ by $5abc$ and $-ac^2x^4$ by $-ax^3$.

7. Divide $-9a^2bc-12ab^2c+15abc^2$ by $-3abc$.

8. From £y take x pence.

---

9. If I walk a miles in b days, what is my rate per day?

10. How far can I walk in m hours if I walk n miles per hour?

11. In x years I shall be y years old: how old am I now?

12. How old is a man who will be x years old in 9 years?

13. Solve the equation $28-4x=5x+10$.

14. Solve the equation $x-3\frac{1}{2}=3\frac{1}{2}+15x$.

15. Solve the equations:

$$(a)\quad \begin{aligned} 2x+3y&=25,\\ 3x+2y&=25. \end{aligned} \qquad\qquad (b)\quad \begin{aligned} 2x&=9-3y,\\ 5y&=24-6x. \end{aligned}$$

16. What number is that the third part of which exceeds its fourth part by 16?

---

17. When 5 is subtracted from a number, $\frac{2}{3}$ of the remainder is 40: find the number.

18. A's age is double of B's, and B's is triple of C's, and the sum of all their ages is 140 years: find the age of each.

19. A farmer has two flocks of sheep, each containing the same number; from one of them he sells 39 and from the other 93, and finds that one flock now contains twice as many sheep as the other: how many sheep did each flock originally contain?

20. After spending £10 more than a third of his yearly income, a person found that he had £15 more than half of it remaining: find his income.

21. Solve the equations:

$$(a)\quad \frac{x}{3}+5=\frac{2y}{3}, \qquad\qquad (b)\quad \frac{x+y}{3}=x-7,$$

$$y-x=\frac{x}{3}. \qquad\qquad\qquad \frac{x}{3}=y+1.$$

22. Subtract $3\{x+2y-\frac{2}{3}(y-4)\}-4(3x-y+2)$
from $3x+2-[7y-4x-\{4(x-2)-3(2y+1)\}]$.

23. Find two numbers the sum of which is 40 and the difference 16.

24. Find two numbers such that, if $\frac{2}{3}$ of the smaller be added to $\frac{1}{4}$ of the larger, the sum shall be 7, and, if $\frac{1}{3}$ of the larger be taken from the smaller, the remainder shall be 2.

---

25. If $a=2$, $b=5$, and $c=3$, find the value of a^3b^2c.

26. Find the values of x and y in the following equations:

$$(a)\quad \frac{1}{8}x+\frac{1}{9}y=42, \qquad\qquad (b)\quad 2x-\frac{y-3}{5}=\frac{5x-2}{2},$$

$$\frac{1}{9}x+\frac{1}{8}y=43. \qquad\qquad 2y-\frac{x-5}{3}=\frac{7y-7}{2}.$$

27. Find the fraction that is equal to $\frac{3}{7}$ when 10 is added to its numerator, and to $\frac{1}{3}$ when 4 is subtracted from its denominator.

28. There were x apples in a box; y apples were taken out and the remainder divided among 5 boys: how many did each boy get?

29. I was x years old 7 years ago: write down an expression for my age y years ago.

30. If x lbs. of sugar cost y pence, what should be paid for 1 lb. and for z lbs.? also, how much of the sugar can be bought for 2 shillings?

31. Asked how much money I had, I replied: "Three-fourths of it exceeds two-thirds of it by £4": how much had I?

32. The sum of three consecutive numbers is 114: find the numbers.

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33. The sum of three consecutive odd numbers is 69: find the numbers.

34. From a cask $\frac{9}{14}$ths full of wine I draw 7 gallons, and then find that the cask is half full: what is the capacity of the cask?

35. If £50 will buy 7 cows and 4 sheep, or 6 cows and 7 sheep, find the value of each animal.

36. Two pieces of cloth differ in length by 8 yards, and half the sum of their lengths is equal to twice the difference of their lengths: how long is each piece?

37. A farm consists of arable land let at 20 shillings per acre and pasture let at 30 shillings per acre, the total rent being £350; when the rent of the arable land is reduced 5 shillings per acre and that of the pasture 8 shillings per acre, the total rent is reduced by £90: how many acres are there in the farm?

38. A man is 72 years old and his wife 63; when they were married he was half as old again as she: how long have they been married?

39. If A can do a piece of work in x hours and B in y hours, find (1) what part of it each can do in 1 hour, (2) what part of it they can do jointly in 1 hour, and (3) the time in which they can do the work, working together.

40. Write down three consecutive numbers, of which a is the middle one, and find their sum.

---

41. If 2 be added to the numerator of a fraction the result is $\frac{1}{2}$, and, if 4 be taken from the denominator of the fraction, the result is the same: find the fraction.

42. A man walks a hours at b miles an hour: how long will it take him to return at c miles an hour?

43. Solve the equation:

$$x-\frac{3}{5}-\frac{5(x-2)}{4}=\frac{3}{2}\left(x-\frac{1}{10}\right).$$

44. A person spends 9*s.* in buying apples at the rate of 7*d.* a dozen and oranges at the rate of 20 a shilling; if he had bought two-thirds as many apples and twice as many oranges, he would have had to pay 13*s.* 4*d.*: how many of each did he buy?

45. If 10 yards of silk and 7 yards of satin cost £5. 6*s.* 4*d.* and 3 yards of the satin cost as much as 4 yards of the silk, find the price of a yard of each material.

46. A clergyman had £5. 10*s.* to distribute among a certain number of old men and women; he found that, if he gave 3*s.* to each, he would be 1*s.* out of pocket; but that, if he gave each man 2*s.* 2*d.* and each woman 3*s.* 6*d.*, he would have 6*d.* to spare: find how many men and how many women there were.

47. A tradesman sends in a bill for £12, part of which is for labour and the rest for materials; if the charge for labour had been twice what it was and the charge for materials one-third of what it was, the amount of the bill would still have been £12: what was the charge for labour and what for materials?

48. A man bought oranges at 6*d.* a dozen and an equal number at 9*d.* a score; he sold them at 9*d.* a dozen and made a profit of 5*s.* 6*d.*: how many oranges did he buy?

---

49. Simplify $\frac{22{\cdot}4}{{\cdot}25}+\frac{250}{{\cdot}8}+\frac{1{\cdot}2}{{\cdot}0075}$.

50. Reduce $\dfrac{\frac{2}{5}+\frac{7}{40}}{\frac{3}{4}-\frac{13}{50}}\times\dfrac{\frac{7}{8}-\frac{21}{200}}{\frac{41}{50}-\frac{3}{16}}$ to a recurring decimal.

51. Write down the value of $(3a-2b)^2$ and of $(497)^2$.

52. If the cost of 24·8 kilograms of sugar is 224·2 francs, find the cost of 74·7 kilograms at the same rate. (Work by logarithms.)

53. If 1·375 tons of coal cost £0·985, what should be paid for 78·7 tons at the same rate? (Work by logarithms.)

54. What decimal of 8 bus. is equivalent to 3 pks. 1 gal. 2 qts. 1 pt.?

55. Find the value of x in the equation:

$$x - \frac{2x - 0{\cdot}3}{0{\cdot}7} = \frac{5 - x}{0{\cdot}35}.$$

56. *A*'s age is *a* years and *B*'s *b* years: what was the sum of their ages *x* years ago?

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57. Reduce $\dfrac{13\left(\frac{6}{13} - \frac{25}{78}\right) - 21\left(\frac{1}{3} - \frac{1}{4}\right)}{13\left(\frac{1}{3} + \frac{1}{7}\right) + 5\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{7}\right)}$ to a decimal.

58. If a metre be equal to 39·371 inches, express a kilometre as a three-place decimal of a mile. (Work by logarithms and check by arithmetical methods.)

59. Taking the seconds' pendulum as 1·087 yards long, find to three places of decimals its length in metres, reckoning a metre as being equal to 3·281 feet. (Work by logarithms and check by arithmetical methods.)

60. The sum of two fractions is $1\frac{1}{12}$, and 6 times the smaller exceeds twice the larger by $\frac{1}{2}$: what are the fractions?

61. Find the simple interest on £*a* for *b* months at *x* per cent. per annum.

62. For one-third of a mile a cable was laid overland, $\frac{3}{13}$ of it was suspended in the water, and $\frac{13}{17}$ of it rested on the bed of the sea: what was the length of the cable?

63. *A* can do a piece of work in *x* hours and *B* in *y* hours: how long will they take to do it, working together?

64. Express 0·00756 in standard form.

---

65. If 74·9 metres of an iron rod of equal thickness throughout weigh 45·78 kilograms, find what length of it would weigh 57·31 kilograms. (Work by logarithms.)

66. Simplify $\frac{6}{11}$ of $3\frac{5}{16}$ of $\dfrac{5\frac{5}{14} - 2\frac{2}{9}}{6\frac{7}{12} - 3\frac{3}{7}}$; and express the area of a section of land measuring 80 feet by 99 feet as a fraction of an acre.

67. Find the value of $\dfrac{(1{\cdot}005 + {\cdot}201) \times (1{\cdot}005 - {\cdot}201)}{1{\cdot}005 \times {\cdot}201}$.

68. Find the value of 327 cwt. 2 qrs. $17\frac{1}{2}$ lbs. at £2. 14*s*. 8*d*. per cwt., (1) by decimalising each of the quantities and finding the product correct to the third place, and (2) by logarithms.

69. *A*'s income is equal to $\frac{3}{4}$ of *B*'s, *A*'s expenditure is equal to $\frac{1}{2}$ of *B*'s income, and *B*'s expenditure is equal to the whole of *A*'s income: show that each has the same sum left.

70. How many pence are equal in value to £$a + b$ halfcrowns $+ c$ florins.

71. A boy walks a miles in b hours: how long does it take him to walk x miles?

72. Find the value of x in the equation:

$$\frac{b}{a}-\frac{dx}{c}=\frac{ax}{b}-\frac{c}{d}.$$

73. 7 lbs. of sugar and 8 lbs. of tea cost 13*s.* 9*d.*, and 9 lbs. of the same sugar and 10 lbs. of the same tea cost 17*s.* 3*d.*: find the prices per lb. of the tea and sugar.

74. If $\frac{1}{18}$ of a certain sum of money by added to $\frac{1}{5}$ of the same sum, the total is found to be less than $\frac{1}{2}$ of the sum by $5\frac{1}{2}$*d.*: find the sum.

75. Express $17\frac{1}{2}$ miles in kilometres, taking a kilometre as being equal to 1093·6 yards.

76. A man and a boy can do in 12 days a piece of work that would be done in $1\frac{7}{23}$ days by 8 boys and 10 men: find how long it would take (1) one man, and (2) one boy, to do it.

77. 30 tons of goods are to be carried in carts and waggons, and it is found that this will require 18 carts and 12 waggons or else 27 carts and 8 waggons: how much can each cart and each waggon carry?

78. Find by logarithms the value of $\sqrt{(16{\cdot}22)^2+(17{\cdot}02)^2}$.

79. The average readings of the thermometer for five days of the week were 58·7, 45·9, 60·7, 41·4, and 53·2: represent these readings graphically.

80. In the following table are given in millions the receipts of the railways in the United Kingdom for the years named: represent them graphically.

1870	1880	1890	1900	1904
43·4	62·9	76·5	98·8	103·8

81. Find the logarithms of each of the following:

$5{\cdot}204\times10^5,\ 5{\cdot}204\times10^3,\ 5{\cdot}204\times10^0,\ 5{\cdot}204\times10^{-1},\ 5{\cdot}204\times10^{-3}.$

82. Express in kilograms per hectare a rainfall of 12 centimetres.

83. Find the value of x in the equation:

$$\tfrac{1}{3}(0{\cdot}75-x)+\tfrac{1}{5}(0{\cdot}47+2x)=(3-\tfrac{1}{15})x.$$

84. A coach travels between two places in 7 hours; if its speed were increased by 2 miles an hour, it would take $5\frac{1}{4}$ hours for the journey: find the distance between the places.

85. A manufacturer makes kettles of the following sizes and prices:

Capacity in quarts	2	6	12
Price - - -	1*s.* 6*d.*	3*s.* 10*d.*	5*s.* 6*d.*

Represent these facts by means of a graph, and estimate from it the probable prices of kettles holding 5 quarts and 9 quarts respectively, and the probable capacity of one costing 5*s.* 2*d.*

86. In the following table are given the percentages of wheat and flour imported into the United Kingdom from British Dominions and Foreign Countries respectively:

Years - - -	1899	1900	1901	1902	1903
British Dominions	21	12·3	19·3	23·6	27
Foreign Countries	79	87·7	80·7	76·4	73

Exhibit these results graphically.

87. (*a*) Express a as a percentage of b. (*b*) A man paid £a for a horse, which he afterwards sold for £b $(a > b)$: find his loss per cent.

88. The product of two numbers is ·15252; if we increase one of them by ·02, the product becomes ·1525446: find the numbers.

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89. I have a certain amount of money with which to buy a certain number of nuts; I find that, if I buy at the rate of 40 a penny, I shall spend 5*d.* too much, and, if at 50 a penny, 10*d.* too little: find how much money I have to spend.

90. A cyclist starts running at 200 yards a minute, and a minute later another cyclist sets out from the same place after him at the rate of 300 yards a minute: draw a graph showing how long the second has to run to overtake the first, and how far they are then from the starting place.

91. An elastic spring is observed to be stretched by weights as follows:

Weight in grams -	10	40	60	80	100	120
Length in cms. -	·9	3·6	5·6	7·2	8·8	10·8

Represent these results by means of a graph. The graph is seen not to pass through the points (60, 5·6) and (100, 8·8). What does this appear to indicate on the part of the observer? What seems to be the law connecting extension with weight? What would you do to test the accuracy of your conclusion?

92. Write the following numbers in standard form, and find their logarithms: 6570000, ·000378.

93. If $a = 2·73$, $b = 1·93$, and $c = 1·387$, find the value of x in the expression:

$$x = \frac{a^2b^2}{\sqrt[3]{c}}.$$

94. A man runs a miles at x miles an hour, b miles at y miles an hour, and c miles at z miles an hour: how long is he on the run?

95. The product of two numbers is 144; if each of the numbers were increased by 2, their product would be 200: find the numbers.

96. Table showing the quantity of wool produced in Australasia for the year 1904:

	lbs.		lbs.
Victoria	75,786,176	Westralia	12,213,635
N. S. Wales	220,502,805	Tasmania	11,412,921
Queensland	46,150,979	New Zealand	149,692,709
South Australia	34,299,232		

Express these results to the nearest tenth of a million pounds and represent them graphically.

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97. A rectangular sheet of paper is 2 cms. longer than wide and its area is 323 square cms.: find its dimensions.

98. The bill of a company at a hotel amounted to £4. 16*s.*; two of the company were not allowed to pay, and the rest found that each of their shares of the bill was 4 shillings more than if all the company had paid: find the number of the company.

99. A cubic centimetre of mercury weighs 13·59 grams; the weight of mercury in a tube 20 cms. long is 79·25 grams: find the capacity of the tube.

100. A certain insurance office gives assurance of £100 at death for the following yearly premiums:

Age of insurer	25	35	45	55
Premium	£2. 6*s.* 6*d.*	£2. 18*s.* 2*d.*	£3. 16*s.* 4*d.*	£5. 13*s.* 8*d.*

Express shillings and pence as decimals of £, plot the points and draw a curve through them, and estimate from it the approximate premium for each of the following ages: 30, 40, 50, 60.

101. Table showing the length of a degree of longitude at every tenth parallel:

Parallel.	Length in miles of a degree of longitude.	Parallel.	Length in miles of a degree of longitude.
0°	69·17	50°	44·55
10°	68·13	60°	34·67
20°	65·03	70°	23·88
30°	59·96	80°	12·13
40°	53·06	90°	0·00

Exhibit these results graphically, and estimate approximately the length of a degree of longitude at latitude 46°.

102. Cwts. of butter and cheese exported from New Zealand in the years named:

	1885	1890	1895	1900	1905
Butter -	24,923	34,816	66,283	161,792	342,853
Cheese -	15,245	40,451	79,650	98,001	82,421

Represent these results graphically.

103. The following results, correct to tenths, are obtained from the weighing of different volumes of the same substance: plot a graph to represent the relation between the volume and the weight of the substance, and find from it the weight of 4 c.c. and the volume of 2·9 grams:

Volume in c.c. -	1·2	2·8	4·1	5·3
Weight in grams	1·0	2·3	3·4	4·4

104. Construct a table of coordinates for the equation $7y=5x$ and draw the graph.

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105. Two couriers, A and B, leave the same place and travel in the same direction; A starts c hours later than B; A travels a miles an hour and B travels b miles an hour: when will A overtake B?

106. A cyclist travels at p miles an hour: how far from home may he ride in order that by returning at the rate of q miles an hour he may be home again in r hours?

107. Two men start at the same time to travel from A to B, a distance of 36 miles; one goes 3 miles an hour more than the other and arrives at B an hour earlier: at what rate does each man travel? Prove graphically.

108. If one part of £400 is put out at 4 per cent. per annum and the other part at 5 per cent. per annum and the yearly income obtained from the investment is £18. 15*s*., what are the parts?

109. The sum of two numbers is 40 and the sum of their squares is 802: find the numbers.

110. The population P (millions) of a certain country is ascertained at the beginning of each of these years to be:

Year -	1830	1835	1840	1850	1860	1865	1870	1880	1890	1895
P -	20	22·1	23·5	29·0	34·2	38·2	41·0	49·4	57·7	63·3

Taking the time in years from 1830, show the relation of P and time by a curve passing fairly evenly through the plotted points, and find from it the probable population in 1848.

111. Plot the graph of the equation $y=\frac{15}{22}x$, and from it determine the rate per hour of a train that travels at the rate of 44 feet per second.

112. The circumference of a circle being taken as $3\frac{1}{7}$ times the diameter, plot the graph that will give the circumference for any diameter up to 30 feet. What is the equation of this graph?

MISCELLANEOUS EXAMPLES. II.

1. 2·35 inches of wire weigh 18·8 grains: find the weight of a mile of such wire.

2. If a piece of wire 6 inches long weighs 2 grains, what will be the length of a ton of the same wire?

3. One pound avoirdupois weighs 7000 grains: express 1·37 ounces in grains.

4. If a body moving at a uniform rate travels over 1·375 miles in ·34375 hour, how far will it travel in 2·379 hours?

5. How many cubic yards of road metal would be required to cover, six inches thick, a road one chain wide and fifty miles long?

6. One person bought three-sevenths of a share in a mining company and another bought four-ninths of a share in the same company: if they paid £220 between them, find how much each of them paid.

7. How long will it take to fill a reservoir 3 miles long, one mile broad, and 20 feet deep, with a pipe which discharges into it 200 cubic feet of water per minute?

8. One person bought five-elevenths of a share in a company and another bought seven-thirteenths of a share in the same company: if the two paid £71, how much did each pay?

9. Find the value of $\frac{49{\cdot}231 \times {\cdot}00728}{{\cdot}0000789}$ to four places of decimals.

10. A clock was 7 minutes 3·9 seconds fast at noon on January 1st, 1866: what was its error at noon on January 1st, 1867, supposing it to have lost uniformly ·37 second a day?

11. The length of a piece of ground is 58·19 feet and its breadth is 43·79 feet: what will it cost at £1. 19*s*. $9\frac{3}{4}$*d*. per square foot?

12. Find by logarithms the area of the surface of a cube containing 27·35 cubic inches. Also find the area of the surface of a sphere having the same cubic content.

13. A train travelling at a uniform rate covers 60 miles in an hour and a half: draw the graph of its motion, and from it write down the time it takes to travel 16 miles and how far it has travelled in 57 minutes.

14. Two persons set out from two distant places at the same time to meet each other; when one had travelled five-elevenths of the distance and the other seven-thirteenths of it they were still a mile apart: find the distance between the two places.

15. The bricks used in building a house cost £708. 2*s.* 6*d.* at £2. 15*s.* a thousand: how much more would the bricks have cost at £3. 2*s.* 6*d.* a thousand?

16. A clock is 1 minute 13 seconds slow at noon on January 1st: how much must it gain daily that it may be 17·5 seconds fast at noon on July 1st?

17. Find the value of $\cdot 125\dot{1}2\dot{5}$ of 999 tons.

18. Of the revenue of a country one-fourteenth is expended in the cost of collection, four-sevenths in the interest of the debt, one-fifth in the maintenance of the army, two-fifteenths in the payment of the civil servants, and the remainder £27319. 3*s.* 4*d.* is available for miscellaneous expenditure: what is the whole revenue of the country?

19. A room is 20 ft. long, 16 ft. wide, and 12 ft. high: what length of tape will reach from one of the lower corners to the opposite upper corner?

20. If 15 metres cost 45·3 francs, find how many yards can be bought for £151, given 1 metre = 39·37 inches and 5 francs = 4 shillings.

21. Of two squares of carpet one measures 44 feet more in perimeter, and 187 square feet more in area, than the other: find the sizes of the squares.

22. Two passengers have between them 345 lbs. of luggage and pay on their excess luggage 4*s.* 2*d.* and 6*s.* 8*d.* respectively: if the luggage had belonged to one of them, the excess charge would have been 15*s.*: find how much free luggage is allowed to each passenger.

23. An estate pays 4*s.* 4½*d.* in the pound: what is the loss on a claim of £865. 4*s.*?

24. What sum will amount to £188. 2*s.* 6*d.* in 18 months at 5 per cent. per annum, simple interest?

25. The rates are struck on 80 per cent. of a rental of £1850 at 9*d.* in the pound: find the amount of the rate.

26. If 40 shillings weigh 7 oz. 5 dwts. and contain 10 dwts. 21 grs. of alloy, how much pure silver is there in one shilling?

27. The annual premium on a fire insurance is £13. 5*s.* 3*d.*, and the duty charged on it is £19. 17*s.* 10½*d.*: how much per cent. of the premium is the duty?

28. If a certain MS. fills 6 printed sheets, each containing 32 pages and each page 24 lines with 9 words in each line, how many lines of 12 words each must there be in a page that another similar MS. twice as long may fill 3 sheets, each having 48 pages?

29. If a cubic foot of marble weighs 2·716 times as much as a cubic foot of water, find the weight of a block 9 ft. 6 inches long, 2 ft. 3 inches broad, and 2 feet thick, supposing a cubic foot of water to weigh 1000 ounces.

30. Divide £1908. 18*s.* among *A*, *B*, and *C*, so that *A*'s share may be to *B*'s share as 3 to 5, and *B*'s share to *C*'s share as 10 to 11.

31. Sugar is bought at £25 a ton, and refining costs £1. 1*s.* 8*d.* per cwt.: if the sugar is sold at 5½*d.* per lb., what is the gain per cent.?

32. Find the difference between $\frac{3}{11}$ of £78. 16*s.* 2½*d.* and $\frac{£35.\ 14s.\ 8\frac{1}{4}d.}{1\frac{2}{3}}$.

33. A person buys equal quantities of oranges at 2 and 3 a penny respectively and then mixes them: how many must he sell for 5*s.* so as neither to gain nor lose?

34. If a person spent £50 more than $\frac{19}{20}$ of a fixed income in a certain year, £35 less than $\frac{14}{15}$ of it in the next year, and $\frac{11}{12}$ of it in the following year, and his savings then amounted to £705, what was his income?

35. Two men carry a weight of 3 cwt. 16 lbs. between them and their loads are to each other as 6 to 5: by how much does the one load exceed the other?

36. Supposing a cubic inch of gold to weigh 10 ozs. 3 dwts. 14 grs., find how much gold will be required to make an acre of gold leaf the two hundred and eighty-two thousandth of an inch in thickness.

37. $\frac{3}{4}(\frac{5}{6}-\frac{4}{5})$ is subtracted from $\frac{9}{13}(\frac{1}{5}+\frac{1}{8})$, and the remainder is divided by two-thirds of the sum of one-half, one-quarter, and one-fifth: express the quotient in its lowest terms.

38. Each link of an ordinary surveyor's chain is too long by the sixteenth of an inch: in what distance, as measured by such a chain, would the error amount to 50 feet?

39. A gun is fired at intervals of one minute: at what intervals will a person hear the report, who is approaching the gun at the rate of 24 miles an hour, supposing the velocity of sound to be 1020·8 feet in a second?

40. Extract the square root of ·00998001.

41. What will £4250 amount to in $2\frac{1}{2}$ years at 5 per cent. per annum compound interest?

42. Sugar is raised in price 50 per cent.: by how much per cent. must I reduce my consumption so as not to increase my expenditure?

43. A plate of metal is 106·58 inches long, 14·6 inches wide, and 2 inches thick: if it were melted and cast into an exact cube, what would be the edge of the cube? (Work by logarithms.)

44. A postman, who receives 15*s.* a week, is fined 1*s.* 6*d.* if he comes late: if at the end of 13 weeks he receives £8. 15*s.* 6*d.*, how often was he late?

45. If money is worth $3\frac{3}{4}$ per cent., what should be the price of the 3 per cent. consols?

46. Of a new company 2562 shares at £20 each are taken: what must be the call per share to raise £8967?

47. *A*'s income consists of £330 paid at Lady-day and £220 at Michaelmas: on the £330 he pays income-tax at the rate of 5*d.* in the pound; and on his whole income he pays 9*d.* in the pound: find what he pays on the £220.

48. The diagonal of a square court is 90 feet: find the cost of paving the court at 2*s.* per 9 square yds.

49. *A* earns £1 in 3 days, *A* and *B* earn £9 in 12 days, and *B* and *C* earn £35. 10*s.* in 42 days: compare the rates of wages of *A*, *B*, and *C*.

50. If a man rows 3 miles per hour against a stream running $2\frac{1}{2}$ miles per hour, how far can he row in $3\frac{1}{2}$ hours with the stream?

51. Find the radius of the earth in miles if its circumference is 40 million metres. (1 inch = 2·54 cms.; take $\pi = \frac{355}{113}$.)

52. A merchant buys wines at 16*s.* 6*d.* and 10*s.* 6*d.* per gallon and mixes them in the proportion of 5 at the latter price to 7 at the former price: at what price per gallon must he sell the mixture to gain 25 per cent.?

53. *A* wins a race of 100 yards from *B* by 17 yards and from *C* by 8 yards: how many yards' start ought *C* to give *B* in 200 yards that they may run a dead heat?

54. A person invests £2852 in shares at 115, paying £5 per share annual dividend: he sells out at 125 and invests in the 3 per cent. consols at 93: find the change in his income.

55. What fraction of $\frac{3}{29}$ of £3. 19*s.* 9*d.* is equal to $\frac{2}{5}$ of £1. 2*s.*?

56. A bookseller sells 3000 copies of a book at 41*s.* each and takes in exchange a bill at six months: how much would he gain or lose by selling them for cash at 39*s.* each, money being at 5 per cent.?

57. An ordinary surveyor's chain is too long by 5·5 inches: what will be the real area of a block of 640 acres measured by such a chain?

58. $\frac{1}{13}\left(\frac{7}{9} - \frac{4}{7}\right)$ is subtracted from $\left(\frac{4}{3} - \frac{9}{7}\right)$ and the remainder is divided by one-seventh of the sum of one-third, one-twelfth, and one-fifteenth: express the quotient in its lowest terms.

59. How many postage stamps ·93 inch long and ·75 inch broad will cover an area 310 ft. long by 100 ft. broad?

60. A vessel steams 16 knots per hour: how many metres per second is this equivalent to, a metre being equal to 3·2809 ft. and a knot to 6086·5 ft.?

61. A tile of copper 1 ft. long, 9 inches wide, and $\frac{5}{8}$ths of an inch thick, is rolled into a plate 6 ft. long and 4 ft. wide: find the thickness of the plate.

62. A ladder 41 ft. 1 inch long is placed against a wall with its foot 12 ft. 11 inches out from the wall: how high will it reach up the wall?

63. An acre is equal to ·40467 hectare and a pound sterling to 25·25 francs: if an estate measuring 1927 hectares is sold for 10,100,000 francs, what is the price per acre in £. *s.* *d.*?

64. A reservoir leaks uniformly ·025 pint in every second: how many gallons will leak out between noon on the 1st January of the present year (1907) and midnight on the 31st December?

65. Assuming the pressure of the atmosphere to be 14·75 lbs. on every square inch of surface, find the whole pressure on the surface of a cube the edge of which is 6 ft.

66. Supposing the value of the copper coinage reduced so that £1 contained 1000 farthings, find the loss which would result in a year in the tolls of a bridge over which 1500 passed daily, paying a halfpenny each.

67. A person buys oranges at eleven pence a dozen and sells them at thirteen for one shilling: how many must he sell to make a profit of £100?

68. Find the number of yards of fencing required to enclose a square paddock of 40 acres.

69. If *A*'s income be 150 per cent. more than *B*'s, how much per cent. is *B*'s income less than *A*'s?

70. *A* and *B*, who live in different towns, are rated on the same amount at 2*s.* and 2*s.* 3*d.* in the £ respectively; and *B* pays £3 more than *A*: find the amount upon which they are rated.

71. A person sells out of the $3\frac{1}{2}$ per cents. at $92\frac{3}{4}$ and realises £18550: he invests $\frac{2}{5}$ of this sum in the 4 per cents. at 96 and the remainder in the 3 per cents. at 90: find the alteration in his income.

72. If 6 men and 2 boys can reap 13 acres in 2 days and 7 men and 5 boys can reap 33 acres in 4 days, how long will it take 2 men and 2 boys to reap 10 acres?

73. If 16 darics make 17 guineas, 19 guineas make 24 pistoles, and 31 pistoles make 38 sequins, how many sequins are there in 1581 darics?

74. Find the value of $\frac{1\cdot17}{\cdot016}$ of 2*s.* 8*d.* $+\,7\cdot2\dot{6}$ of 5*s.* $-\,\frac{2\cdot\dot{7}}{\cdot4\dot{4}\dot{5}}$ of £1. 4*s.* 6*d.*

75. The hands of a watch are observed to come together at intervals of $65\frac{1}{2}$ minutes true time: find its rate.

76. If a pound of silver be coined into 66 shillings and a pound of gold into £46. 14*s.* 6*d.*, compare the weight of a sovereign with that of a shilling.

77. Find the value of $\frac{\cdot2}{37}$ of $\frac{3\cdot3\dot{9}-1\cdot\dot{1}\dot{8}}{5\cdot\dot{2}9\dot{7}+3\cdot02\dot{6}\dot{6}}$ of £264. 13*s.* $6\frac{1}{4}$*d.*

78. A person who has £3645 can make £6 a year more by investing it in the 4 per cents. at $102\frac{1}{2}$ than in the $3\frac{1}{2}$ per cents: what is the price of the latter stock?

79. The sides of a rectangle are 16 feet and 10 feet respectively: find by logarithms the length of a diagonal of a square which is equal in area to the rectangle.

80. Weighed in air at a temperature of 62° F., one gallon of water weighs 10 lbs,, and a cubic centimetre of water weighs ·99755 gram: find, correct to three decimal places, the number of litres in a gallon. (1 gram = 15·4323 grains.)

81. The ages of *A* and *B* are 15 years 4 months and 17 years 7 months respectively: how much would each get if £19. 15*s.* were divided between them proportionally to their ages?

82. An army besieging a town consists of 3116 men and 124 officers: when the place is captured each man gets as his share of the booty £32. 11*s.* $7\frac{1}{2}$*d.*, and each officer gets on an average 3 times as much as a private: find the total booty taken.

83. What diminution per cent. is made in a man's income when the income-tax is raised from 7*d.* to 9*d.* in the pound?

84. A tradesman owes the following sums: £56. 17*s.* $8\frac{1}{2}$*d.*, £43. 12*s.* 5*d.*, £176. 4*s.* 10*d.*, and £16. 9*s.* $3\frac{1}{2}$*d.* He has owing to him in good debts £49. 8*s.* 6*d.*, £64. 6*s.* 9*d.*, and £13. 14*s.* 8*d.*; and in bad debts £52. 6*s.* 8*d.*, £95. 17*s.* 6*d.*, and £43. 3*s.* 4*d.*, for which he receives respectively 7*s.* 6*d.*, 3*s.* 4*d.*, and 15*s.* in the £: is he able to pay his debts; and, if not, how much in the pound can he pay?

85. The gross rental of an estate is £6240; the income-tax is 7*d.* in the £, and the expense of collecting is $\frac{1}{20}$ of the remainder: find the net rental.

86. If 1 man can transcribe 3 pages of a certain book in an hour and a half, how long will it take 3 men, working only half as rapidly, to transcribe 36 pages of another book the pages of which contain one-third as much again as the former?

87. If a cubic foot of gold be made to cover 432,000,000 square inches, find the thickness of the coating.

88. By selling cloth at 8*s.* 9*d.* a yard $\frac{1}{20}$ of the cost is gained: what would be the percentage of loss or of gain if it were sold at 7*s.* 9*d.* per yard?

89. If 2 horses be worth as much as 5 oxen, and 3 oxen as much as 16 sheep, what is the value of a horse, if a sheep is worth £3. 15*s.*?

90. A pyramidal roof 16 feet high, standing on a square base the side of which is 24 feet, is covered with sheet-lead $\frac{1}{16}$ of an inch thick: find the weight of the lead, supposing a cubic inch of it to weigh 7 ounces avoirdupois.

91. A paper was worked by 2500 pupils, of whom $\frac{1}{5}$ were girls and the rest boys; 5 per cent. of the boys and 40 per cent. of the girls failed: what percentage of the number examined passed?

92. *A* gives *B* 10 yards' start and *C* 15 yards in a race of 100 yards: how much should *B* give *C* in 150 yards?

93. If 4 men working 7 hours a day do a piece of work in 5 days and 1 hour, how many hours a day must 6 men work to do twice as much in 6 days and 4 hours, allowing for a half-holiday on one of the 6 days?

94. If 5 town workmen can do as much work as 9 country workmen, what would be the difference to a contractor between employing 15 town men, or 10 town men and 12 country men, to do a piece of work which the 15 workmen could do in 10 days, town men being paid 7*s.* 6*d.* and country men 4*s.* 6*d.* a day?

95. Fifty thousand soldiers are to be raised in four provinces which contain respectively 175,002, 82,434, 22,116, and 20,448 men fit for service: find the number of men each province should supply.

96. Messengers travelling 10 miles an hour are sent out every 6 minutes to meet a person approaching at 5 miles an hour: at what intervals will they meet him?

97. The price of 4 sheep and 9 lambs is £4. 10*s.*, and that of 8 sheep and 3 lambs is £6. 10*s.*: find the value of a sheep and of a lamb.

98. I invest £5187. 10*s.* in the 3 per cents. at 83, but afterwards transfer $\frac{3}{5}$ of this sum to the 4 per cents. at 96: how is my income affected?

99. If a reduction of 20 per cent. were made in the price of eggs, it would enable a person to obtain 54 more for a guinea: find the present price.

100. A rectangular sheet of tin of uniform thickness is 85 centimetres wide and 2·7 metres long, and weighs 536 grams: find the thickness if 1 cubic centimetre of tin weighs 7·3 grams.

101. If a cubic yard of road metal will metal six superficial yards, how many cubic yards will be required to metal twenty-one miles of road fifteen feet wide?

102. A roll of carpet 32 yards long and 2 feet 8 inches wide is used to make a square of carpet, which is laid evenly in the middle of a rectangular room the dimensions of which are 24 feet and 18½ feet: determine the area of the uncovered part of the floor and the distance of the carpet from the sides and the ends of the room.

103. Each member of a society subscribes as many pence as there are members in the society, and the total subscription is £29. 8*s.*: find how many members are in the society.

104. A person measured the distance between two places and found it 30 miles; but he afterwards found that his chain, instead of being 22 yards long, was an inch and a half too short: find the true distance between the two places.

105. A pipe delivers water at the rate of three pints in 35 seconds: how long will it take to fill a reservoir holding a million gallons?

106. Two clocks are together at noon on March 1st: if one gains uniformly three minutes a day, and the other 36 seconds a day, when will the difference between them be three hours and a quarter?

107. A man is 72 years old, and his wife 63; when they were married, the husband was half as old again as his wife: how long have they been married?

108. Ten per cent. of the gross profits of a station is paid for management, and 1*s.* 4*d.* in the £ income-tax is deducted from the remainder: if the net income is £2646, find the profits.

109. At noon a train leaves *A* for *B*, a distance of 120 miles, and travels at the rate of 25 miles an hour; an hour afterwards another train leaves *A* for *B*, and travels at the rate of 40 miles an hour: determine graphically where and when the faster train overtakes the slower, and when the slower is 8 miles ahead of the faster.

110. A certain sum of money is lent for 219 days: the rate per cent. per annum is 4¼ and the interest is £55. 15*s.* 7½*d.*: find the sum.

111. A person left £100 per annum for ever to a school in his parish: what sum must have been invested to purchase this annuity in the 3½ per cents. at 93⅓?

112. If 4 fowls are worth 3 ducks and 7 ducks are worth 2 geese and 9 geese are worth 7 turkeys, what is the price of a fowl, the price of a turkey being 18 shillings?

113. A cubical cistern open at the top costs 15 guineas to line with lead at 1*s.* 9*d.* per square foot: how many cubic feet does the cistern contain?

114. A person bought £1 worth of steel pens at 1*s.* 4*d.* per box, each containing one gross, and retailed them at 2½*d.* per dozen: how much per cent. did he gain on his outlay?

115. If the cost of building a wall 135 yds. 2 ft. long and 1 ft. 9 inches thick be £17. 16*s.* 1½*d.*, what would be the cost of building one three times the height, 86 yds. 1 ft. long, and 1 ft. 10 inches thick, other things being equal?

116. The price of 14 sheep and 11 pigs is £51. 10*s.* and that of 8 sheep and 33 pigs is £69. 10*s.* : find the value of a sheep and of a pig.

117. What is the ratio of the interest per annum to the principal, if £765. 12*s.* 6*d.* amount to £878. 3*s.* $5\frac{1}{4}$*d.* in 4 years 73 days?

118. Find the value of $\frac{\cdot 0807-(\cdot 27-\cdot 1902)}{3\cdot 7-3(1\cdot 3-\cdot 0671)}$ of 19*s.* 6*d.*

119. A cubic inch of water weighs 252·5 grains and a gallon contains 277·274 cubic inches: express in tons and decimals of a ton the weight of 1000 gallons of water.

120. Multiply 1 mile 7 furlongs 3 chains by 76543.

121. A student has 2 hours for exercise; he starts off on a coach that travels at 10 miles an hour, dismounts after a time and walks back at the rate of 4 miles an hour: what is the greatest distance he can travel by coach and keep within his time?

122. What is the diagonal of a cube the square on the edge of which is 5625 feet?

123. Divide 218 tons 6 cwt. 2 qrs. by $28\frac{2}{7}$.

124. If, after all taxes are paid, the net income of an estate is £534. 15*s.* and the gross income is £570. 8*s.*, to how much in the £ do the taxes amount?

125. A printing-machine turns out 3500 sheets per hour: if its speed be decreased in the ratio of 7 to 4, how many sheets can be printed in a day of 12 hours?

126. Two equal wine-glasses are filled with mixtures of spirit and water in the ratios of 1 of spirit to 4 of water and 1 of spirit to 5 of water: if the contents be poured into a tumbler, find the strength of the mixture.

127. By disposing of goods for £364 a man loses $\frac{4\frac{1}{2}}{50}$ of what he gave for them: what should be the selling price to gain $\frac{1\frac{3}{4}}{25}$ of their cost price?

128. Gunpowder is composed of 33 parts of nitre, 7 of charcoal, and 5 of sulphur; what weight of each of these components is contained in a ton of gunpowder?

129. Divide £709. 8*s.* 3*d.* amongst 4 persons in the proportion of the fractions $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$.

130. After paying two taxes, one at the rate of 3*d.* in the £ and the other amounting to $\frac{1}{40}$ of my gross income, I have a net income of £1193. 10*s.*: find my gross income.

131. By selling an article for £82 a man gained $\frac{3}{20}$ of the cost: find the cost and the gain per cent.

132. How much is $\frac{1}{4}\left(\frac{4\frac{7}{9}\text{ of }6\frac{2}{3}}{7\frac{26}{27}}\right)\times\frac{3\frac{2}{5}-3\frac{1}{6}}{3\frac{3}{5}+2\frac{1}{6}}$ of £182. 7*s.* 5*d.*?

133. Find in kilograms the weight of an iron girder which is 5·4 metres long, ·54 metre wide, and ·45 metre thick, having given that a cubic centimetre of iron weighs 7·76 grams.

134. A ladder 28·6 feet long reaches to a window that is as high above the ground as the foot of the ladder is distant from the house: find by logarithms the height of the window.

135. Divide £5·64 into two parts such that a fourth of one part shall exceed the eighth of the other by £·42.

136. A log of kauri pine measures in section 5 ft. 3 inches by 4 ft. 6 inches: what length of it must be cut off to obtain 630 cubic feet of timber?

137. What sum of money has the same ratio to £1. 2*s.* 6*d.* that 11·55 yards have to $3\frac{3}{8}$ yards?

138. A contractor has to ballast 45 miles of railway in 12 weeks, and at the end of 3 weeks finds that with 42 men he has completed only 5 miles: how many additional men must he engage in order to finish his undertaking within contract time?

139. Seventy-five per cent. of the area of a farm is arable; of the remainder 85 per cent. is pasture, and the rest is waste; the area of the waste is 3 acres 20 poles: find the area of the farm.

140. The difference between the interest and the discount on a certain sum for 9 months at $3\frac{1}{3}$ per cent. was 10*s.* 6*d.*: find the sum.

141. A legacy of £1500 is payable at the end of 2 years: find its present value, allowing compound interest at 5 per cent. per annum.

142. An analysis of a solid weighing 1·56 grams yields ·991 gram of substance *A*, ·441 gram of substance *B*, and ·128 gram of substance *C*: calculate its percentage composition. (Use logarithms.)

143. A sum of money put out at compound interest amounts in 1 year to £4200 and in 3 years to £4630. 10*s.*: find the rate per cent. and the sum.

144. Standard gold contains $\frac{1}{12}$ of alloy: if 40 lbs. troy of standard gold be minted into 1869 sovereigns, how many grains of pure gold are in a sovereign?

145. A policeman runs after a boy, who starts a little distance ahead; three of the policeman's steps are equal to five of the boy's, but the boy takes three steps while the policeman takes two: shew whether or not the policeman will catch the boy.

146. Find the value of $\left(4-\cfrac{1}{2-\cfrac{1}{3+\frac{1}{2}}}\right)$ of 2 qrs. 16 lbs.

147. A tradesman's prices are 50 per cent. above cost price: if he allows a customer 8 per cent. on his bill, what profit does he make?

148. At what times between 5 and 6 o'clock are the hands of a clock at right angles to each other?

149. Find to the nearest second how long it would take a man walking at the rate of $3\frac{1}{2}$ miles an hour to go round a square paddock that contains 64 acres.

150. A rod appears to measure 6 ft. 10 ins., but $\frac{1}{12}$ inch has been worn off from one end and $\frac{1}{15}$ inch from the other end of the yard measure used: find the real length of the rod.

151. A tradesman's prices are $2\frac{1}{4}$ times the cost price: if he allows a customer 8 per cent. on his bill, what profit does he make?

152. There is a number of which the twelfth part exceeds the fifteenth part by 24: find the number.

153. A solid rectangular block of copper 3 ft. long, 1 ft. 10 inches wide, and 2 ft. deep, is melted down and made into a wire the sectional area of which is $\frac{1}{15}$ of a square inch: find the length of the wire.

154. Find the square root of $\frac{15129}{182329}$.

155. Divide £5. 4*s*. 6*d*. among *A*, *B*, *C*, so that for each fivepence received by *A*, *B* may receive sixpence, and for each threepence received by *B*, *C* may receive fourpence.

156. *B* and *C* can do a piece of work in $1\frac{7}{8}$ hours, *C* and *A* can do it in 2 hours, and *A* and *B* can do it in $2\frac{8}{11}$ hours: in what time can each of them separately do the work?

157. If a cubic yard of clay makes 460 bricks, each 9 inches long, $4\frac{1}{2}$ inches wide, and $2\frac{1}{2}$ inches deep, how much per cent. does clay contract in the baking?

158. (*a*) Express as a decimal of £3 the difference between ·4625 of 16 shillings and $\cdot 113\dot{6}$ of £8. 5*s*. 0*d*. (*b*) Find by logarithms the square root of the product of $3\cdot\dot{2}\dot{7}$ and $7\cdot 2\dot{3}$.

159. A person pays a county rate which exceeds his income-tax of 7*d*. in the pound by £45; and, after paying both taxes, he has a net income of £463. 10*s*. a year: find his gross income.

160. If one pound of tea be worth 50 oranges, and 70 oranges be worth 84 lemons, and 20 lemons cost one shilling, what is the price of the tea?

161. The seconds' pendulum being 1·0872 yds. long, express its length in metres to 4 places of decimals, the metre being 39·37 inches.

162. A boy is told to find ·61 of a sum of money; he finds $\cdot\dot{6}\dot{1}$ instead: if his error amounts to 1*s*. $3\frac{1}{4}$*d*., what is the sum of money?

163. A vessel contains 210 gallons of spirit; 45 gallons are drawn off and an equal quantity of water added: if this operation is performed a second and a third time, how many gallons of pure spirit remain?

164. A sum of money is to be divided amongst *A*, *B*, and *C*; *A* is to have $\frac{1}{4}$ of the whole, *B* $\frac{3}{7}$ of the remainder, and the rest is to go to *C*; *C*'s share proves to be £210: what is the share of each of the others?

165. What sum of money due 1 year hence is equivalent to £100 due now together with £300 due 2 years hence, interest being reckoned at 5 per cent. per annum?

166. The difference in the income arising from investing a certain sum in the 3 per cents. at 92, or in the $3\frac{1}{2}$ per cents. at $107\frac{1}{4}$, is 10*d*.; find the sum.

167. A bankrupt has good debts to the amount of £456. 18*s*. 3*d*.; and the following bad debts, £360. 7*s*. 6*d*., £120. 13*s*., and £21. 4*s*., for which he receives 4*s*., 5*s*., and 10*s*. in the £ respectively: if his own liabilities amount to £4558, how much can he pay in the £?

168. A goods train is 14 miles ahead of an express which travels at the rate of 50 miles an hour; and the goods train is run into in 35 minutes: find the rate at which the goods train is travelling.

169. A piece of work can be done by A alone in 3 hours, by B alone in 4 hours, and by C alone in 6 hours; B and C begin the work, and, after $\frac{3}{4}$ of an hour, are joined by A, and half-an-hour before the work is finished B leaves off: find the time in which the work is done.

170. A path of uniform width runs round the interior of a rectangular courtyard 46 feet long and 36 feet wide, and encloses a lawn the area of which is 133 square yards, 3 square feet: find the width of the path.

171. Find in lbs. (to two places of decimals) the weight of copper in 100 miles of wire the diameter of which is ·001 inch, assuming that 1 lb. of copper occupies 3·1 cubic inches. (Take $\pi=\frac{22}{7}$.)

172. Express a pressure of 1 ton per sq. foot in kilograms per sq. metre, a kilogram being $2\frac{1}{5}$ lbs., and a metre 39·37 inches.

173. A bankrupt, whose property is worth £468. 15*s.*, owes 4 creditors £1500: the claims of two of the creditors are £263. 1*s.* 8*d.* and £403. 12*s.* 4*d.* respectively: if to one of the other two he owes half as much again as to the other, find the dividend each of these receives.

174. If on a straight road stones be placed at intervals of 50 ft., and a measuring tape of 15 yards be used, how often will the end of the tape coincide with the end of a stone in a distance of one mile?

175. A man bought a certain number of sheep for £97. 10*s.* at £1. 17*s.* 6*d.* a head; some of them died, and he then sold the rest at £2. 4*s.* 6*d.* each, and gained on the whole £2. 12*s.* 6*d.*: find how many died.

176. A ship costing £9600 is wrecked: what does a person lose who possessed $\frac{3}{16}$ of it, the ship being insured for $\frac{19}{20}$ of its real value?

177. A gave $\frac{7}{10}$ of what he had in his purse to B; B gave to C $\frac{15}{28}$ of what he received from A; C gave to D $\frac{8}{9}$ of what he received from B: if D received 14*s.* 1*d.*, how much had A in his purse at first?

178. If coffee and chicory cost £8. 10*s.* and £2. 10*s.* per cwt. respectively, what is the proportion of coffee and chicory in a mixture of which 7 lbs. are worth 7*s.* 6*d.*?

179. If one gallon of spirit which contains 11 per cent. of water is added to 3 gallons containing 7 per cent. of water, and to this mixture is added half a gallon of water, what percentage of water will there be in the whole mixture?

180. Given that the area of a circle $=\frac{7}{88}(\text{circumference})^2$, find to the nearest yard the difference between the lengths of the boundaries of a circle and a square, each containing 10 acres.

181. The value of ·1625 of a field at £17 per acre is £40. 1*s.* $1\frac{1}{2}$*d.*: how much is the field worth? If the area of the field is $\cdot080\dot{5}$ of the area of the farm to which it belongs, what is the area of the farm?

182. If 10 inches = 254 millimetres, find the percentage of error when we reckon 20 metres = 1 chain.

183. *A* and his boy *B* undertake with the help of a fellow-workman *C* to dig a garden for £6; they finish the job in 8 days; *A* alone could do the work in 16 days, and *C* alone could do it in 20 days: how ought the contract price to be divided among *A*, *B*, and *C*?

184. Divide £4050 among *A*, *B*, *C*, *D*, and *E*, so that *A*'s share may be to *B*'s as 1 to 2, *C*'s share to *B*'s as 5 to 4, *D*'s share to *C*'s as 6 to 5, and *E*'s share to *D*'s as 4 to 3.

185. A district containing 8460 acres is rated on an annual value of £10575: if a rate of 9*d.* in the pound be raised, what is the average charge per acre?

186. The lengths of the sides of a triangular plot of ground are 20 chains, 24 chains, and 20 chains: find in acres the area of the plot.

187. A schoolroom is to be built to accommodate 70 children, so as to allow $8\frac{1}{2}$ square feet of floor and $110\frac{1}{2}$ cubic feet of space for each child: if the room be 34 feet long, what must be its breadth and height?

188. An inch of rain has fallen on 3 acres of land: find the total weight of the water, assuming that a cubic yard of water weighs 15 cwt.

189. Find the cost of the bricks needed for the building of a wall 30 yards long, 6 feet high, and $13\frac{1}{2}$ inches thick, having given that 1000 bricks cost 25 shillings, and that each brick fills up a space 9 inches long, $4\frac{1}{2}$ inches wide, and 3 inches deep.

190. *A* and *B* start at the same time from two points 62 miles 7 furlongs apart, *A* travelling at the rate of 3 miles every 55 minutes, and *B* $2\frac{1}{2}$ miles every 38 minutes: when and where will they meet?

191. For what sum, to the nearest penny, must a vessel worth £10,000 be insured at £3. 12*s.* 6*d.* per cent. that in case of loss both her value and the premium of insurance may be recovered?

192. On what date will £547. 10*s.*, lent on the 1st January, 1907, amount to £572. 2*s.* at 5 per cent. per annum, simple interest?

193. On the sale of a section of land for £105 there was a loss of 16 per cent.: what would the selling price have been if 16 per cent. had been gained?

194. How much water must be mixed with spirit at 22 shillings per gallon to fill a cask of 121 gallons, so that a gallon may be worth 18 shillings?

195. A train 88 yards long overtook a man walking along the line at the rate of 4 miles per hour, and passed him completely in 10 seconds; it afterwards overtook another man and passed him in 9 seconds: at what rate per hour was the second man walking?

196. $\frac{1}{6}$ of the trees in a garden are 10 ft. high, $\frac{2}{3}$ of the remainder are 12 ft., and the remaining 15 are 16 ft. high: how many trees are there in the garden, and how far would they reach if placed end to end?

197. Draw to the scale of ·1 inch to a chain a plan of a triangular field each side of which is 32 chains, and find the area of the field.

198. Find correct to four places of decimals the value of

$$\frac{3{\cdot}14159265 \times 792{\cdot}679283}{2{\cdot}7182818}.$$

199. How many yards of fencing will be required for a square field having an area of 314 acs. 2 ro. 19 po. 1¼ yds.?

200. A cubical block contains 2048383 cubic inches: how much will it cost to paint its surface at 9*d*. a square yard?

201. A train is exactly 27 minutes in passing through a tunnel the length of which is 11220 metres: supposing a metre to be 39·39 inches, find the speed of the train in miles per hour.

202. Express the quotient of £6. 16*s*. 6*d*. divided by $\left(\dfrac{\frac{1}{2}\text{ of }\frac{7}{9}}{\frac{2}{3}\text{ of }\frac{1}{4}}-\dfrac{\frac{3}{5}\text{ of }\frac{1}{6}}{\frac{5}{8}\text{ of }\frac{1}{7}}\right)$ as the decimal of £6. 15*s*.

203. Find the value of $\left(\dfrac{\cdot0019}{3\cdot1\dot{6}}\text{ of }\dfrac{4\cdot\dot{4}}{\cdot0005}\right)\div\left(\dfrac{8\cdot\dot{8}}{7}\text{ of }\dfrac{4}{5\cdot625}\right)$.

204. A merchant bought a number of bottles of claret at 35*s*. per dozen, and another number at 21*s*. per dozen; he mixed the two wines and sold at 30*s*. per dozen without gaining or losing by the transaction: in what proportion were the two wines mixed?

205. A dealer has two sorts of tea, which he sells at the same price, thereby making a profit of 5 per cent. on the one and 20 per cent. on the other: determine his total profit per cent. if he sells 7 lbs. of the former to every 2 lbs. of the latter.

206. A cistern which is 8 ft. 4 ins. long and as deep as it is wide, contains 150 cub. ft. 900 cub. ins.: what would be the expense of lining its bottom and sides with lead at 2*s*. 5*d*. per sq. ft.?

207. A dealer sells an article at £32. 10*s*. per ton, 15 per cent. of that sum being profit: what price per ton must be charged in order that 35 per cent. of it may be profit?

208. The length of a hall is 3 times the breadth; the cost of whitewashing the ceiling at 5⅓*d*. per sq. yd. is £4. 12*s*. $7\cdot\dot{1}$*d*., and the cost of papering the four walls at 1*s*. 9*d*. per sq. yd. is £35: find the height of the wall.

209. A section of land contains a quarter of an acre and has a frontage of 66 ft.: it is laid down in grass with the exception of a path 5 ft. wide which runs round it: find the area of the grass plot.

210. A sum of £1032. 6*s*. 2*d*. is to be divided amongst an equal number of men, women, and boys—in all 78; each woman is to receive 5 times as much as a boy, and each man as much as a woman and two boys together: find how much each will receive.

211. 45,114,180,144 sovereigns are divided into 6008 parcels, half of which are twice as large as the remainder: how many sovereigns are there in a parcel of each size?

212. *A* cycles at 8 miles an hour, but rests half an hour at the end of each hour; *B*, starting 2 hours after *A* and travelling uniformly without resting, overtakes *A* at 23 miles from the starting point: draw graphs of their motions, and determine from them to the nearest mile the rate at which *B* travels per hour.

213. A shopkeeper marks his goods at a price from which he can deduct 5 per cent. for prompt payment, and still make a profit of 14 per cent. on what the goods cost him: find what he paid for an article marked £6. 6*s*.

214. A person wishes to make an income of £100 per annum by investing £2666. 13*s*. 4*d*. in the 3 per cent. consols: find what price they must be quoted at when he buys.

215. Find to the nearest tenth of a square cm. the area of a triangle the sides of which are 29, 17, and 19 cms. long.

216. Telegraph posts are placed along a railway at intervals of 70 yards; and an engine-driver finds that he passes three of these intervals in 11 seconds: find the rate of the train in miles per hour.

217. If the price of gold be £3. 18*s*. per ounce troy and 1 lb. avoirdupois contains $14\frac{7}{12}$ ounces troy, find the number of pounds avoirdupois in £423423 worth of gold.

218. What decimal part of a square mile is equal to 48 acres?

219. A man saves £100 every year, and invests it at the end of the year at 5 per cent. per annum compound interest: find the amount of his savings at the end of the 4th year.

220. From the following notes, draw a plan and find the area of the field (lengths in links):

	To *B*	
	1200	To *H* 160
To *G* 280	1120	
To *F* 300	960	
	940	To *E* 400
To *D* 760	760	
	200	To *C* 30
	From *A*	

221. A boat's crew rowed 4 miles in 20 minutes against a stream flowing 3 miles an hour: how long would they take to row the same distance with the stream?

222. Two cyclists, *A* and *B*, start from a certain place to travel along the same road to a place 45 miles distant; *A* starts at noon and travels at the rate of 10 miles an hour and *B* starts at 2 p.m. and travels at the rate of 20 miles an hour: find graphically when and where *B* overtakes *A*, and when they are 5 miles apart after *B* starts.

223. In a circular riding school 118 feet in diameter, a circular ride 10 feet wide is to be made just within the outer edge of the building: find the cost of making it at $3\frac{1}{2}$*d*. per square yard.

224. A dealer purchases a liquid at 4*s*. a gallon and dilutes it with so much water that, when he sells the compound at 3*s*. a gallon, he gains 20 per cent. on his outlay: how much water is there in every gallon of the compound?

225. Find the solid and the superficial contents of a cylindrical ring the thickness of which is 9 inches and the inner diameter 32 inches. $(\pi=3\frac{1}{7}.)$

226. The sides of the base of a triangular prism are 17, 25, and 28 feet, and its length is 20 feet: find its volume.

227. A vessel contains 10 gallons of wine; one gallon is drawn off and a gallon of water substituted: if this operation is repeated ten times in succession, how much wine remains in the vessel? (Work by logarithms.)

228. If a cubic centimetre of gold weighs 19·3 grams, how many pounds avoirdupois does a cubic foot of it weigh, given that 1 gram = 15·4323 grains and 1 centimetre = ·3937 inch?

229. Find the value of $\dfrac{\sqrt[3]{28\cdot934443}}{\sqrt[3]{3\cdot375}}$ and $\sqrt{(3\frac{1}{2})^5}$.

230. In what time will £225 amount to £251. 5*s.* $5\frac{169}{256}$*d.* at $3\frac{3}{4}$ per cent. per annum compound interest?

231. A square reservoir covers an area of 160 acres and is 70 ft. deep: if after a dry month the volume of water is found to be 97574400 cubic feet, how many feet has it sunk from the top of the reservoir?

232. What number is that from which if $\frac{7}{11}$ of it be taken there remains 36?

233. A pyramid is 24 feet high and stands on a square base the side of which is 64 feet: find its slant surface and its volume.

234. A person bought 40 lbs. of sugar of two different sorts for £1. 5*s.* 4*d.*; the better sort cost 10*d.* per lb. and the worse 7*d.*: find how much he bought of each sort.

235. How many revolutions per minute will be made by a carriage wheel 4·55 feet in diameter when the carriage is going at the rate of 5·2 miles in 1·25 hours? ($\pi = \frac{22}{7}$.)

236. If I take 3 steps in every 2 yards and take $1\frac{1}{2}$ seconds to make 4 steps, how far can I walk in an hour?

237. *A* saves £36 a year and *B* £60 a year: plot two graphs to the same axes, showing *A*'s and *B*'s accumulated savings up to the end of 20 years, and find from them the difference of the savings at the end of the 10th, 15th, 17th, and 20th years.

238. Two towns are ten miles apart; from each of them a person starts at noon to walk to the other; one person takes 5 paces while the other takes 6, and the length of the former's pace to that of the latter is as 4 to 5: find where they will meet.

239. A man has £11. 12*s.* made up of an equal number of the following coins—five-shilling pieces, half-crowns, shillings, sixpences, fourpenny pieces, threepenny pieces, and pennies: how many of each coin has he?

240. A rate of 1*s.* 8*d.* in the £ is struck upon 90 per cent. of the rent, and the rent and rates together form what is called the estimated annual value: find the rateable value of a property the estimated annual value of which is £559.

241. How many yards of matting $2\cdot\dot{4}$ feet broad will cover a floor $27\cdot\dot{3}$ ft. long and $20\cdot1\dot{6}$ ft. broad?

242. If 3 men, or 4 women, or 5 boys, or 6 girls can perform a piece of work in 60 days, how long will it take 1 man, 2 women, 3 boys, and 4 girls to do it all working together?

243. The area of a rectangular field is 39 acres 32 poles and one side is double the other: find its sides.

244. An army lost 18 per cent. of its strength by disease and desertion and then lost 14 per cent. of the remainder in battle; the number then remaining was 84624: of how many did the army originally consist?

245. What must be the market value of 3 per cent. stock in order that, after deducting an income-tax of 10*d.* in the pound, it may yield $3\frac{1}{2}$ per cent. interest?

246. A canal 10 miles long is 8 yards wide at the top, 6 yards wide at the bottom, and 5 feet deep: how long did it take 800 men to make it, each man removing on an average 15 cubic yards per day of 24 hours?

247. A detachment of cavalry consisted of 520 men after engaging in two battles, in the first of which it lost 1 man in every 25 and in the second $\frac{5}{18}$ of the remainder: of how many men did the troop originally consist?

248. A rectangular grass plot, the length of whose sides are in the ratio of 5 to 6, costs £16. 11*s.* $6\frac{3}{8}$*d.* for turfing at 5*d.* per square yard: find the length of its sides.

249. If $1\frac{3}{5}$ yards of cloth are worth $\frac{5}{12}$ of a bushel of corn and 12 yards of cloth will fetch $4\frac{1}{2}$ dollars, what is the value of 5 quarters of corn, a dollar being equal to 4*s.* 2*d.*?

250. A prize of £2770 was divided amongst three persons in the proportion of the fractions $\frac{3}{4}$, $\frac{14}{15}$, and $\frac{5}{8}$: find the share of each person.

251. One side of a square field measures 35·25 chains: determine the length of its diagonal.

252. A person paid £18. 15*s.* for a year's income-tax; but afterwards the government raised the tax by 9*d.* in the pound and then he paid £52. 10*s.* as income-tax: find his income and the rate in the pound at which he was first taxed.

253. How many yards of carpet 2 ft. 3 inches wide will be required to cover the floor of a room 35 ft. 3 inches long by 22 ft. 6 inches wide, and what will be its cost at 6*s.* 9*d.* a yard?

254. The diagonal of a cube is $21\sqrt{3}$ feet: find the cost of painting the surface of the cube at 4*d.* per square yard.

255. Find the volume of a pyramid the height of which is 12 inches and the base an equilateral triangle each side of which is 10 inches.

256. A plank 15 feet long rests vertically against a perpendicular wall: how far must the bottom end be pulled out to lower the top end 3 feet?

257. A pendulum swings through an angle of 30° and the end describes an arc of $13\frac{2}{21}$ inches: find the length of the pendulum. (Take $\pi = 3\frac{1}{7}$.)

258. The height of a room is 14 ft. 7 inches and its content is 9803 cubic feet 416 cubic inches, and it is known to be twice as long as it is broad: find its length.

259. If 8 men working 8 hours a day, or 12 boys working 6 hours a day, can do a piece of work in 21 days, how long would 16 men and 30 boys take to do a piece of work twice as large as the former, all working 7 hours a day?

260. A tradesman sells an article for £12, thereby losing 4 per cent., and recovers his loss by selling another article for £8: find his gain per cent. on the second article.

261. Three miners own a claim, *A* having one-third, *B* one-fourth, and *C* the remainder: *A*, however, abandons the ground, and his share is divided between his partners in the proportion of their interests: what are now the shares respectively of *B* and *C*?

262. If I invest £5747 in the New Zealand 5 per cents. at $102\frac{1}{2}$ and sell out at $103\frac{1}{4}$, after having received a half-year's dividend, what sum shall I have gained, the brokerage both on the purchase and the sale of the stock being $\frac{1}{8}$ per cent.?

263. What is that number which, when added to three-fourths of itself, amounts to ten millions fifty-four thousand and two?

264. Find the least weight which can be expressed exactly either in pounds avoirdupois or in pounds troy.

265. Having given that a gallon of water weighs 10 lbs. and a cubic foot of water weighs 1000 ozs., find to what depth a rectangular cistern, 16 feet 8 inches long and 8 feet 3 inches broad, is filled when it contains 11000 gallons.

266. A section of a stream is 10 feet wide and 10 inches deep; the mean flow of the water through the section is 3 miles an hour: taking 25 gallons to equal 4 cubic feet, find how many gallons of water flow through the section in 24 hours.

267. A clock, set to correct time at 8 o'clock on Monday morning, is found to have gained 3 minutes 20 seconds at 5 o'clock (true time) on Tuesday afternoon: supposing the clock to go uniformly, what will be the correct time when it strikes 11 on Sunday forenoon?

268. The diagonal of a trapezium is 34·4 chains, and the two perpendiculars from the opposite corners are 7·5 and 9·4 chains respectively: find the area in acres.

269. The outer diameter of a water-pipe is 2 feet, the inner diameter is 1 foot 8 inches, and the length of the pipe is 40 feet: find the number of solid inches of metal used in the construction of the pipe. (Take $\pi = 3{\cdot}1416$.)

270. The solid content of a triangular prism is $4{\cdot}1\dot{6}$ cubic feet and the sides of the triangular end are 2 feet 1 inch, 1 foot 8 inches, and 1 foot 3 inches: find the length of the prism.

271. Two trains start from the same place at the same time and in the same direction, and at the end of $2\frac{1}{2}$ hours one is 30 miles ahead of the other: if they had gone in opposite directions they would have been 130 miles apart in the same time: find the speed of each per hour.

272. A person has some 3 per cent. stock, which yields him an income of £30 a year; he sells out, and with part of the proceeds is able to buy as much 5 per cent. stock at 105 as will yield him the same income as before: if he now has £110 remaining, what was the price of the original stock?

273. A block of wood 3 feet thick and of specific gravity ·85 rests on a square base the side of which is 5 feet: find the weight of the block. (1 cub. ft. of water $= 62{\cdot}3$ lbs.)

274. A block of rock of specific gravity 3·76 weighs a ton and a half: find its volume.

275. A reservoir is supplied by one pipe and emptied by another: the supply-pipe would fill it in 4 hours and the waste pipe empty it in 9 hours: if both run together, how long will it take to fill the reservoir?

276. Two perpendiculars *CP*, *DQ*, are drawn to the side *AB* of the quadrilateral field *ABDC*, and measure respectively 352 links and 595 links; also *AP* is 110 and *AQ* 745 links, and the whole line *AB* is 1110 links: find the area.

277. If oranges be bought at the rate of 16 for a shilling, how many must be sold for a shilling to gain 60 per cent.?

278. A squatter engaged 20 shearers, who could have shorn his flock in 24 days; but after 8 days' work 4 of the shearers left him: how long will the remainder take to finish the work?

279. A rectangular block of stone, having a square base each side of which is $28\frac{1}{2}$ inches, contains $62137\frac{1}{2}$ cubic inches: find its height.

280. A contractor engaged to make 1920 yards of railway in 120 days, and for that purpose engaged 160 men; but after 24 days' work he found that he had executed only 240 yards: how many additional men must he engage to finish the work in the stipulated time?

281. A person makes arrangements to lay by £150 at the end of every year: find how much he will possess at the end of 4 years, reckoning compound interest at 5 per cent. per annum.

282. A ring, the internal and external radii of which are 3·81 cms. and 5·79 cms. respectively, is cut from a uniform sheet of iron; the weight of the ring is 21·89 grams: find its thickness, supposing 1 cubic centimetre of iron to weigh 7·8 grams. (Take $\pi = 3\frac{1}{7}$.)

283. Taking the pressure of the atmosphere to be 14·73 lbs. per square inch, express this pressure in grams per square centimetre, having given that 1 lb. = 453·6 grams and 1 inch = 2·54 centimetres.

284. If unity be divided into two parts of which the one part is equal to $\frac{3}{4}$ of $\frac{5}{6}$ of $(2\frac{1}{2} - 1\frac{3}{5})$, find the other part.

285. Find the value of $\sqrt{2-\sqrt{2}}$ to five places of decimals.

286. A society having to raise £1507. 10*s*. in a year finds that a monthly contribution of 2*s*. $9\frac{1}{2}$*d*. per head will be sufficient: what is the number of members in the society? If 96 members decline to pay, what additional monthly contribution must be made by the remaining members?

287. What fraction of a crown is equal in value to the difference between seven-eighths of a pound and seven-ninths of a guinea?

288. A hare pursued by a dog was 43 yards before him at starting: how far had the dog to run before he caught the hare, supposing him to have run 7 yards to every 5 that the hare ran? Represent their motions graphically.

289. The number of feet in the distance through which a body falls to the earth is 16 times the square of the number of seconds occupied by the fall: in how many seconds would a body fall 3136 feet?

290. In the following table is given the population per square mile for the countries and years named: represent the results graphically:

	1801.	1821.	1841.	1861.	1881.	1898.
England	155	207	275	347	443	536
Scotland	53	68	86	100	122	143
Ireland	165	212	256	181	161	139

291. The simple interest on a certain sum for 2 years at 7½ per cent. per annum was £160. 10*s.*, and the compound interest on the same sum for the same time and rate was £166·51875: find the sum.

292. How much in English money (correct to pence) should a traveller receive for a bill of 1000 francs when the rate of exchange is 25·2 francs for £1 and ¼ per cent. is charged for brokerage?

293. A debtor paid off one-fifth of his debts, then one-third of what remained, and then one-fourth of what still remained: supposing that he still owed £60, find the total amount of his original debts.

294. A tank can be filled by one pipe in 5 hours and by another in 4 hours: the first pipe is turned on alone until the tank is half full, when the second is turned on also: how long did they take to fill the tank?

295. A square field is bounded by a path 3 yards wide, the field and path together occupying 2½ acres: find the cost of covering the path with gravel at 1*s.* 6*d.* per square yard.

296. How many square feet of canvas will it take to cover the sides and top of a box 6 ft. 8 ins. long, 4 ft. 4 ins. wide, and 3 ft. 7 ins. deep?

297. A nautical mile is greater than an imperial mile by $\cdot 1\dot{5}$ of the latter: how many nautical miles are there in 730000 feet?

298. One liquid contains 22½ per cent. of water and another 27 per cent.; a glass is filled with 5 parts of the one liquid and 7 parts of the other: what percentage of water is in the glass?

299. Find in acres the area of a quadrilateral field of which the diagonal is 1274 links and the perpendicular upon it from the opposite angles 550 and 583 links.

300. A waggon loaded with hay weighs 3 tons 19 cwt. $11\frac{1}{5}$ lbs.: the weight of the waggon alone is 40 per cent. of the weight of the load: if the load is worth £13. 15*s.* 5¼*d.*, find the value of the hay per ton.

301. On what principal would the compound interest for 2 years at 6 per cent. per annum equal the simple interest on £494. 8*s.* for 2 years 6 months at 5 per cent. per annum?

302. The perimeter of one square field is 588 yards and that of another 672 yards: find the perimeter of a third which is equal in area to the two others.

303. Find to the nearest penny the cost of gilding, at 6*d.* per square foot, the slant surface of a cone the altitude of which is 6 feet and the diameter of the circular base is 3 feet. (Take $\pi = 3{\cdot}1416$.)

304. The quotient of an operation in division is 515450 and the divisor is to the quotient as 9 to 65: find the dividend.

305. A gentleman left an estate of 735 acres: if when sold $\frac{1}{5}$ of the land brought £25 per acre, $\frac{1}{3}$ of it £20 per acre, and the remainder £15 per acre, and the proceeds were divided among his three sons in the proportion of 3, 4, and 5, find each son's share.

306. 20000 loads of gravel were to be carted to a given place in 10 days; but it was found that 54 horses had carted only 4500 loads in 6 days: how many more horses would be required to cart the rest in the time left?

307. *A* can mow 5 acres in $9\frac{1}{3}$ days and *B* $4\frac{1}{3}$ acres in 7 days: how long will it take them together to mow $12\frac{1}{8}$ acres, and how much of this will each mow?

308. If the diameter of a globe be 40 inches, what is the area of its convex surface?

309. If the clearing of a paddock containing 29·325 acres cost £75·072, what will the clearing of a similar paddock containing 35·19 acres cost, wages having risen in the proportion of 1·25 to 1?

310. I have a certain sum of money with which to buy a certain number of nuts, and I find that if I buy at the rate of 40 a penny I shall spend 5*d.* too much, and if at the rate of 50 a penny 10*d.* too little: how much have I to spend?

311. A bill drawn 21st December, 1906, at 4 months, was discounted on the 11th of February, 1907: if the true discount at 6 per cent. per annum amounted to £5. 8*s.*, find the amount of the bill.

312. A draper having bought 4000 yards of cloth, sold $\frac{2}{5}$ of it at a loss of 5 per cent., $\frac{1}{4}$ of the remainder at a gain of $7\frac{1}{2}$ per cent., and the rest at a gain of $2\frac{1}{2}$ per cent.: if he gained £2. 1*s.* 8*d.* by the whole transaction, find the cost price of the cloth per yard.

313. Find how much more than $\frac{\cdot0338184}{\cdot03416}$ of $1\cdot1\dot{6}$ of $\cdot6$ of $\cdot5\dot{8}\dot{7}$ of £5. 4*s.* 2*d.* I need to pay a bill of £2. 2*s.* 6*d.*

314. Three men contracted to build a railway 59 miles 36 chains 32 links long, and agreed that each man should construct a length corresponding to the capital he could employ: if *A* placed £6517 in the concern, *B* £7820. 8*s.*, and *C* £9123. 16*s.*, find the length of line each had to make.

315. Two men, *A* and *B*, start to walk a distance of 1000 miles in 1000 hours: on an average *A* rests 9 hours and *B* 7 hours per day: what is the difference between the average speed of *A* and *B* if they both complete the distance in the given time?

316. A rectangular field contains $1\frac{1}{2}$ acres; the distance between opposite corners is $6\frac{1}{2}$ chains: find the length and breadth of the field.

317. A moat 6 feet deep and 18 feet wide surrounds a circular islet 115 feet in diameter: find the quantity of water in the moat, taking 1 cubic foot as measuring $6\frac{1}{4}$ gallons. $(\pi=3\frac{1}{7}.)$

318. The diameters of the bores of two guns are 6 inches and 8 inches respectively: if their weights vary as the cubes of the diameters of their bores and the weight of the smaller gun is 3 tons 15 cwt. 3 qrs. 21 lbs., find the weight of the larger one.

319. A quantity of tea was sold for 4*s.* 2*d.* per pound at a gain of 10 per cent.: if the total gain was £12, what quantity of tea was sold?

320. If a farthing be the interest on a shilling for a calendar month, what is the rate of interest per cent. per annum?

321. Find the price of the 3 per cents. when an investment of £434. 12*s.* 6*d.* produces an income of £14. 5*s.*

322. The discount on a certain sum of money due 5 years hence is £20, and the interest on the same sum for the same time is £25: find the sum and the rate per cent. per annum.

323. An account after a discount of $2\frac{1}{2}$ per cent. is taken off becomes £16. 14*s.* 9*d.*: what was the sum due before the discount was subtracted?

324. If a litre is ·22 gallon, find to the nearest penny in English money the value of a pint of liquid which is worth 10 francs the litre, 1200 francs being equivalent to £49.

325. A school of boys and girls contains 453 children, and the boys are ·525252... of the girls: how many boys are there?

326. The diagonals of a rhombus being 88 and 234 feet, find the area. Also find the length of a side and the height of the rhombus.

327. A conical tent is $7\frac{1}{2}$ feet high and encloses 200 square yards of ground: find how much canvas there is in it.

328. From two villages 40 miles apart two cyclists set out at the same instant, travelling towards each other at the rate of 14 and 11 miles respectively: show graphically when they meet, how far each has then travelled, and when they are first separated by 10 miles.

329. I sold 60 bags of sugar at a profit of 6 per cent. and 30 at a profit of 8 per cent.; but had I sold the whole at a uniform profit of 7 per cent. I should have gained 15 shillings more than I did: find the prime cost per bag.

330. Simplify $\sqrt{\left\{(1\frac{1}{3}+2\frac{2}{7})\left(\frac{5\frac{1}{16}}{4\frac{6}{7}+1\frac{1}{4}}\right)\right\}}\div\frac{3}{4}$.

331. If a circular piece of pavement three feet in diameter costs a guinea, find the cost of paving a circular court one-eighth of a mile in diameter.

332. If the air contains 21 per cent. by volume of oxygen, find the number of cubic feet of oxygen in a room which measures $16\frac{1}{2}$ feet every way.

333. A firm lost in one year 3800 francs: find the loss of each of the two partners, if their shares were to each other as $\frac{3}{4}$ to $\frac{5}{6}$.

334. Three drovers, *A*, *B*, and *C*, hire a pasture for 51 francs 70 centimes for 5 months: *A* puts in 60 cows, *B* 80 cows, and *C* 90 cows: if at the end of 3 months *A* sells $\frac{1}{3}$ of his, *B* $\frac{1}{4}$ of his, and *C* $\frac{1}{5}$ of his, find how much of the rent each ought to pay.

335. The diagonal of a rectangle, the length of which is 45 feet, is 53 feet; find the side of a square of equal area.

336. A boy was given the value of $\frac{6}{7}$ of a certain sum and was asked to find what the sum was; he multiplied the value given by $\frac{6}{7}$ instead of by the reciprocal of $\frac{6}{7}$ and got an answer too small by £32. 10*s.*: what was the certain sum?

337. The area of a rectangular field is $\frac{3}{5}$ of an acre and its length is double its breadth: determine the length of each of its sides. If a pony is tethered to the middle point of one of the longer sides, find what length of tether will enable him to graze over half the field. (Take $\pi = 3\frac{1}{7}$, and work by logs.)

338. Prove that the area of a sector of a circle is equal to half the product of the length of the arc and the radius. If the area is 2240·567 square feet and the radius 33·495, calculate the length of the arc.

339. A rectangular field is p feet long and q feet wide, and from one corner there is cut off a portion in the shape of a right-angled triangle, of which the right angle coincides with the corner of the field, and of which the sides that contain the right angle are each a feet long: find the cost of fencing the field at £x per chain.

340. A rectangular reservoir 184 feet 10 inches long, 63 feet 3 inches broad, and 7 feet 6 inches deep, was full of water at 10 o'clock on Saturday night, and at 11 o'clock a.m. on the following Monday it was found to contain 478170 gallons of water: supposing that a gallon $= 277\frac{1}{4}$ cubic inches and that the decrease was uniform, find to what depth the surface of the water had sunk and at what rate per hour.

341. What is the weight of a canvas tent consisting of a cylindrical lower part 10 feet high and 60 feet in diameter, surmounted by a conical portion the highest point of which is 50 feet from the ground, the weight of a square yard of the canvas being 1 lb. 2 oz. (Take $\pi = 3\frac{1}{7}$.)

342. A gallon of water weighs 10 lbs. and a litre of water weighs 1 kilogramme: having given that 1 litre $=$ 1·76 pints, express 1 lb. as the decimal of a kilogramme.

343. A man passes through his lungs 15 cubic feet of air per hour: supposing that he is shut up in a close room 7 ft. long and 7 ft. wide, and that all the air in the room will have passed once through his lungs in 29 hours 24 minutes, find the height of the room.

344. The area of a semicircle is 13013 square feet: find its total perimeter. (Take $\pi = 3\frac{1}{7}$.)

345. A pyramid of lead is 14 inches high, and stands on a square base the side of which is 6 inches long: how many cubic inches of lead does it contain and how many spherical bullets ·75 inch in diameter can be made out of it? ($\pi = \frac{22}{7}$.)

346. When silk is sold at 19 francs 25 centimes the metre, find the price per yard in shillings and pence, supposing £1 $=$ 25 francs 20 centimes and 1 metre $=$ 39·371 inches.

347. If 24 Englishmen in 72 days of 10 hours each can dig a trench 16 feet deep and 28800 square yards in horizontal section, how many Chinamen in 42 days of 15 hours each can dig a trench 15 feet deep, 90 yards wide, and 400 yards long, the work of an Englishman being to that of a Chinaman as 8 to 5?

348. Find the value of $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}-\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$ to three places of decimals.

349. The circumference of the base of an upright cone being 44 feet and the slant height 10 feet, find the area of the convex surface.

350. If the cost of paving a circular court 42 feet in diameter be £63, find the cost of paving one the diameter of which is 78 feet.

351. Find the diameter and the area of the surface of a sphere the volume of which is 5946 cubic inches. (Take $\pi=3\frac{1}{7}$, and work by logs.)

352. Seventy-four tons of produce can be carried either in 16 carts and 20 waggons, or in 21 carts and 17 waggons: what weight can each cart and each waggon carry?

353. The population of a country consists of p males and q females; of the former x per cent. and of the latter y per cent. can read and write, and of the whole population z per cent. can read and write: write down the equation that expresses this fact.

354. The sum of £15927. 3*s.* 9*d.* was borrowed by the school board of a parish, to be paid off in yearly instalments of $3\frac{1}{2}$ per cent.: if the rate levied to meet the annual payment was $5\frac{1}{4}d.$ in the pound, find the rateable value of the parish.

355. A man bought a horse and carriage for £100, and sold the horse at a gain of 50 per cent. on its prime cost and the carriage at a loss of 25 per cent., thus gaining 5 per cent. on his whole outlay: find the buying and the selling price of the horse and the carriage respectively.

356. How many gallons of spirits at 31 shillings per gallon must be added to 16 gallons at 17 shillings a gallon, and 19 gallons at 27 shillings a gallon, to make a mixture worth 26 shillings per gallon?

357. A merchant sold a horse at a loss of 4 per cent.; but had he sold it for 4 guineas more he would have gained 5 per cent.: find the original cost.

358. A man possessed £2500; he invested part of it in 4 per cent. stock at 96 and the remainder in 6 per cent. stock at 117: if his income was £116. 13*s.* 4*d.*, find how much he invested in each kind of stock.

359. A spherical balloon is to contain 4851 cubic yards of gas: how many yards of silk $\frac{3}{4}$ of a yard wide will be required to make it? ($\pi=3\frac{1}{7}$.)

360. The sum of the ages of A and B is now 60 years, and their ages 10 years ago were as 5 to 3: find their present ages.

361. A house is offered for sale at an increase of 12 per cent. on its cost price; but, not being disposed of, it is now offered for £80 less than before; it being still unsold, this second offer is reduced by $2\frac{1}{2}$ per cent. of itself, and the house is then sold: if a profit of 4 per cent. is made by the sale, determine the original cost of the house.

362. If the velocity of sound be 1100 ft. per second and that of a cannon ball 1000 ft. per second, calculate the distance from the gun of a spot where the report is heard three-fourths of a second before it is struck by the shot.

363. Supposing sound to travel at the rate of 1140 feet per second: if a shot be fired from a ship moving at the rate of 10 miles an hour, how far will the ship have moved before the report is heard at a place $14\frac{1}{4}$ miles off?

364. How many pounds of tobacco at 5*s*. 3*d*. per lb. must a tobacconist mix with 4 lbs. at 6*s*. 6*d*. per lb. that he may sell the mixture at 7*s*. 10*d*. per lb. and gain $\frac{1}{3}$ of his outlay?

365. Suppose a circular pipe, the diameter of which is $1\frac{1}{2}$ inches, will fill a tank in 5 hours: in what time will a similar pipe, the diameter of which is $3\frac{1}{2}$ inches, fill the same tank, if the water flows with the same velocity?

366. If 25 francs 60 centimes be worth 6 thalers 20 silbergroschen, how many francs and centimes is a thaler worth, a thaler being equal to 30 silbergroschen and a franc equal to 100 centimes?

367. A man's income consists of a fixed salary of £510 per annum, of dividends on shares paying 5 per cent. per annum, and of rents: if his dividends form one-sixth of his total income and the rents one-eighth, find the amount of capital he has invested in shares.

368. Cannons are fired at intervals of ten minutes in a town towards which a passenger train is approaching at the rate of 35 miles an hour: if sound travels at 1142 feet per second, find at what intervals the reports will be heard by the passengers.

369. A stone dropped down a shaft falls through a number of feet equal to 16·1 times the square of the number of seconds during which it is falling: after how many seconds will the stone reach the bottom of a shaft 6766·025 feet deep.

370. The length of a pendulum varying as the square of the number of seconds in its time of vibration: if a pendulum whose time of vibration is one second is 39·1386 inches long, what is the length of a pendulum whose time of vibration is $\frac{3}{5}$ of a second?

371. *A* sold to *B*, at a profit of 25 per cent., a section which cost him £180, and *B* sold it to *C*, making a profit of 10 per cent.: if *A* had sold the section directly to *C* at the same price that *C* paid for it, what would he have gained and what would have been his gain per cent.?

372. A river is 12 feet deep and 96 yards wide, and the water flows at the rate of $4\frac{1}{2}$ miles per hour: how many gallons flow past a given point in 5 minutes, given that 100 gallons of water occupy a space of 16 cubic feet?

373. The cost of gravelling a rectangular court-yard, three times as long as broad, at $10\frac{1}{2}$*d*. per square yard, is £7. 1*s*. 2*d*.: find the cost of erecting a fence round it at 4*s*. 6*d*. per rod.

374. A person embarked the whole of his capital, which was originally £1275, in three successive speculations, in the first of which he gained 55 per cent. and in each of the others he lost 20 per cent.: find his whole gain or loss.

375. The velocity acquired by a falling body is proportional to the square root of the height from which it has fallen, and it is 40 feet per second when the body has fallen through 25 feet: what will be the velocity when the body has fallen through 72 feet?

376. A section of land is of the shape of a right-angled triangle; the length of the hypotenuse is one mile, and the length of one of its sides is half a mile: find the area of the section in acres to two places of decimals.

377. How many pounds of powder can be put into a hollow spherical shell the internal diameter of which is 13 inches, the number of cubic inches in one pound of powder being 31·06? (Take $\pi=3\frac{1}{7}$, and work by logs.)

378. Find the length of wire $\frac{1}{32}$ inch in diameter that can be made from a cubical piece of iron each side of which is 3 inches. (Take $\pi=3\cdot1416$.)

379. If a kilogram is equal to $2\frac{1}{5}$ lbs. and a metre is 39·37 inches, express the atmospheric pressure, which is $14\frac{3}{4}$ lbs. per square inch, in kilograms per square metre.

380. To a puncheon of rum, containing 84 gallons, which cost £100. 16*s.*, a dealer added a certain quantity of water and retailed the mixture at 28 shillings a gallon, thereby making a profit of 40 per cent.: what quantity of water did he add?

381. If the mass of the Earth be $\frac{1}{314760}$ of that of the Sun and the mass of Jupiter $\frac{1}{1048}$ of that of the Sun, what ratio does the mass of Jupiter bear to that of the Earth?

382. A tank can be filled by one tap in 2 hours, by a second in 3 hours, and by a third in 4 hours; the first tap is allowed to run for half-an-hour, when the second is turned on, and, at the end of another half hour, the other is turned on: in how many hours will the tank be filled from the time when the first tap was turned on?

383. If under similar conditions 318 litres of nitrogen gas are of the same weight as 125 litres of chlorine gas, 18 litres of chlorine gas of the same weight as 79·5 litres of marsh gas, 15·2 litres of marsh gas of the same weight as 14·4 litres of ammonia gas, 18 litres of ammonia gas of the same weight as 152 litres of hydrogen gas, and 900 litres of hydrogen gas weigh 1247·4 grains, find in lbs. avoirdupois the weight of 12000 litres of nitrogen gas.

384. I have to be at a certain place in a certain time, and I find that if I walk at the rate of 4 miles per hour I shall be five minutes too late, and if at the rate of 5 miles per hour I shall be 10 minutes too soon: what distance have I to go?

385. If mining shares, bought at $25\frac{1}{2}$ per cent. premium, pay $7\frac{1}{2}$ per cent. on the investment, how much per cent. would they pay if bought at $10\frac{1}{2}$ per cent. discount, $\frac{1}{2}$ per cent. in each case being allowed for brokerage?

386. A ball 4 inches in diameter weighs 9 lb: find the diameter of a similar ball that weighs 42 lbs.

387. Of two circular ponds the smaller has a diameter of 30 yards, and the area of the larger is twice that of the smaller: find the diameter of the larger pond.

388. If a metre = 39·36 inches, 1 kilogram = $2\frac{1}{5}$ lbs., and 9° Fahrenheit = 5° Centigrade, and if the work of raising a pound one foot be called a foot-pound, and that of raising a kilogram one metre be called a kilogram-metre, express 902 foot-pounds per degree Fahrenheit in kilogram-metres per degree Centigrade.

389. It is known that the squares of the periodic times of the planets are proportional to the cubes of their mean distances from the Sun, and that the periodic times of the Earth and Jupiter are respectively 1 year and 11·862 years: hence calculate to three decimal places the mean distance of Jupiter from the Sun, that of the Earth being taken as unity.

390. It is known that the volume of a gas is proportional directly to its absolute temperature and inversely to its pressure, and it is observed that a mass of air occupies one cubic foot when its absolute temperature is 300°, and its pressure is 15 lbs. per square inch: find its volume when its absolute temperature is 450°, and its pressure 18 lbs. per square inch.

391. The capital of a firm has been contributed by three partners as follows: *A* £1200, *B* £1800, *C* £3000; and it is agreed that the active partners, *A* and *B*, shall receive 20 per cent. and 13⅓ per cent. respectively of the gross profits for their services in the business: if the gross profits amount to £1500, what will be the share of each of the partners?

392. A rectangular hall is twice as long as broad and twice as broad as high: find its dimensions, having given that the cost of covering the floor with linoleum at 4*s*. 3*d*. a square yard came to £42. 10*s*. Find also the cost of plastering the walls and ceiling at 3*s*. 8*d*. per square yard, after deducting one-sixth of the area of the walls as an allowance for doors and windows.

393. Air consists of 20·9 per cent. oxygen and 79·1 per cent. nitrogen by volume, and 23·1 per cent. oxygen and 76·9 per cent. nitrogen by weight: compare the weights of oxygen and nitrogen with an equal volume of air taken as unity.

394. A person holding £14350 New Zealand 4½ per cent. stock sells out at 105, and with the proceeds purchases Exchequer bills, bearing interest at the rate of 8*s*. 9*d*. per cent. monthly, when the bills are at a premium of £2. 10*s*. per cent.: find the alteration in his income.

395. Wheat is quoted in London at £2. 15*s*. per quarter and in Antwerp at 26 francs per hectolitre; freight to London is 5 shillings per quarter and to Antwerp 2 francs per hectolitre; and exchange with London is 1½ per cent. and with Antwerp 2½ per cent.: determine to which of these ports a grain dealer should consign his wheat, and what he will thereby gain per quarter. (£1 = 25·26 francs: 1 quarter = 2·908 hectolitres.)

396. If the work of raising a pound one foot be called a foot-pound and the work of raising a kilogram one metre be called a kilogram-metre, express a horse-power, which is 550 foot-pounds per second, in kilogram-metres. (1 metre = 39·37 inches: 1 kilogram = 2·2 lbs.)

397. If cloth be bought at 2*s*. 9*d*. a yard, find what its selling price must be to leave a profit of 30 per cent. after allowing a discount for ready money of sixpence in the pound.

398. When the barometer stands at 30 inches the pressure of the atmosphere is 2120 lbs. per square foot: hence find the pressure in dynes per square centimetre when the height of the barometer is 75 centimetres. (1 foot = 30·48 centimetres: 1 lb. = 453·6 grammes: weight of 1 gramme = 981 dynes.)

399. A person offered a section of land at an upset price which would have yielded him a profit of 24 per cent., but, being obliged to take £70 less than the upset price, he lost 16 per cent.: what did he receive for the section?

400. Brown sells his goods 20 per cent. cheaper than Black Brothers and 20 per cent. dearer than White and Grey: how much would a customer of Black Brothers save by taking £500 worth of goods (1) from Brown and (2) from White and Grey?

401. A spherical ball of lead, 3 inches in diameter, is melted and recast into three spherical balls: the diameters of two of these are 1½ ins. and 2 ins. respectively; determine the diameter of the other ball.

402. *A* and *B* together could do a piece of work in 40 days; after working 10 days they are assisted by *C*, and the work is then finished in 20 days more; *C* does as much work in 2 days as *B* does in 3 days: in how many days could each of them do the work alone?

403. A merchant imports 1335 chests of tea; he sells $\frac{1}{3}$ of them at a profit of 25 per cent., $\frac{1}{5}$ at a profit of 15 per cent., and the remainder at a profit of 5 per cent., his total gain being £1064. 5*s.* 10*d.*: find how much per chest the tea cost him.

404. A sum of money was left to *A*, *B*, and *C* in the proportions of 7, 9, and 5 respectively; *A*'s share invested at 5 per cent. brought in less than *B*'s invested at 4 per cent. by 12 shillings a year: find the total sum left and the amount each received.

405. A man adds annually to his capital £50 more than the eighth part of the original amount, and at the end of 6 years finds that the whole amounts to £11675: what was the original capital?

406. Simplify $\frac{7}{15}\left(\frac{\sqrt{2-\frac{3}{5}}\times\sqrt{3+\frac{4}{7}}}{3\sqrt{2-\frac{3}{5}}\div\sqrt{3+\frac{4}{7}}}\right)$.

407. Two partners, *A* and *B*, started a business, *B* finding half as much again of the capital as *A*; at the end of 8 months *B* withdrew half of his capital, and 2 months later on *A* withdrew quarter of his capital: at the end of the year the profits were found to be £530; to how much of this profit was each man entitled?

408. *A*, *B*, and *C* run a race of 2 miles; *A*'s rate is to *B*'s as 5 to 4, and *B*'s rate to *C*'s as 7 to 5: how many yards start must be given to *C* that he may win the race by 6 yards?

409. *A* runs a 2-mile race with *B* and loses; had he run a third quicker he would have won by 22 yards: compare their rates of running.

410. What must be the dimensions of a rectangular cistern of which the base is square and the depth equal to a quarter of an edge of the base, that it may contain 19321 gallons of water, each gallon containing 278 cubic inches?

411. A reservoir containing 1200 cubic feet of water is emptied by means of a pipe 3 inches in diameter: if the water flows out with a velocity of 2 miles an hour, how long will it take to empty it?

412. A hemispherical punch-bowl is 5 ft. 6 ins. round the brim; supposing it to be half full, how many persons may be helped from it in hemispherical glasses $1\frac{3}{4}$ ins. in diameter?

413. A man rows a certain distance with the current in 3 hours and the same distance against the current in 4 hours: compare the rate of rowing and the rate of the current.

414. A goldsmith receives an order for a gold cup of 40 ounces weight and 20 carats fineness, *i.e.* 20 parts out of 24 pure gold and the remaining 4 alloy of little or no value: he had in stock some gold 18 carats fine and other gold 22 carats fine: find how much of each he must use, and also the proportion of alloy contributed by the 18-carat to that by the 22-carat gold.

415. A dealer bought 3 sheep of equal value; the first was sold for £3. 11*s.* 3*d.* at a loss of 5 per cent., the second at a profit of 10 per cent., and the third at a price which produced a profit of 4 per cent. on the whole transaction: for what did he sell the third sheep, and what was the gain per cent. on that sheep?

416. *A* and *B* row on a river, starting simultaneously from the same point, *A* down stream and *B* up stream; in $3\frac{3}{4}$ minutes they are five furlongs apart; *A* then turns to follow *B*, and at the end of $12\frac{1}{2}$ minutes from that time the boats have together rowed 1 mile 7 furlongs in all: the speeds of *A* and *B* and the stream being constant, at what rate does the river flow?

417. In the Centigrade thermometer the point at which water boils is marked 100° and the freezing point 0°, and in the Fahrenheit thermometer the boiling point is 212° and the freezing point 32°: at what temperature would the two instruments indicate the same number of degrees?

418. The working capacities of three men, *A*, *B*, and *C*, are in the proportion of 2, $1\frac{1}{2}$, and 1; the three work together for 4 days, when *C* leaves, and *A* and *B* continue the work for $1\frac{1}{2}$ days; *B* then leaves and *A* works alone for the remainder of the last day: in what time would *A* and *B*, working without *C*, have completed the same amount of work?

419. The cost of levelling a square exercise ground is £70; and if each side were 3 feet longer the cost would be increased by £7. 3*s.* 6*d.*: find the length of the side of the square and the cost per sq. yd.

420. *A* and *B* receive 1600 cabbages to plant, but of these 10 prove worthless; *A* plants 5 while *B* plants 4, and the rows have the same number of cabbages; after some hours they have not sufficient between them to complete another row, *A* having 45 to spare and *B* 6; *A* then finds that he has planted 3 rows more than *B*: how many did each plant and how many in a row?

421. In a square field is a square pond; the area of that part of the field not occupied by the pond is 2·484 times that of the pond: what is the ratio of a side of the field to a side of the pond?

422. At a siege it was found that a certain length of trench could be dug by the soldiers and navvies in 4 days, but that when only half the navvies were present it took 7 days to dig the same length of trench: what proportion of the work was done by the soldiers?

423. I buy two horses for £110: if I sell the first at a loss of 5 per cent. and the second at a gain of 5 per cent. I shall gain $\frac{5}{11}$ per cent. by the transaction: find the cost price of each horse.

424. A fraudulent tradesman removed from a cask a certain quantity of whisky that contained 60 per cent. of alcohol, replacing it with diluted spirit containing only 8 per cent. of alcohol; the cask was then of 50 per cent. strength: determine what portion of the whole quantity he removed.

425. Two trains, 92 ft. and 84 ft. long respectively, when moving with uniform velocity on parallel lines in opposite directions, pass each other in 1½ seconds, but when moving in the same direction the faster passes the slower in 5 seconds: find their rates of travelling.

426. A wall, 5 times as high as it is broad and 8 times as long as it is high, contains 18225 cubic feet: find the breadth of the wall.

427. A bill upon Amsterdam is bought at 12 florins per £ sterling; the proceeds purchased at Amsterdam bills upon Hamburg at 34¼ florins for 40 marks, and these are sent to Paris and sold at 185 francs per 100 marks: what is the rate of exchange between London and Paris?

428. At present *B*'s age is greater than *A*'s by 3½ per cent. of *A*'s, and in 12 years it will be greater by 2½ per cent. of *A*'s age: find their present ages.

429. In Reaumur's thermometer the freezing point is zero and the boiling point 80°; in Fahrenheit's thermometer the freezing point is 32° and the boiling point 212°: what will Reaumur's thermometer mark when Fahrenheit's marks 47°, and what will Fahrenheit's mark when Reaumur's marks 93°?

430. It is noticed that the water in a reservoir, 38 feet long and 26 feet wide, which is known to leak, sinks one inch in 12 hours; a pipe discharging 60 gallons per minute will fill the reservoir in 45 hours, allowing for the leakage: assuming a cubic foot of water to contain 6·25 gallons, find within an inch the depth of the reservoir.

431. The price of land about London is every year rising 4 per cent. on previous prices, while the purchasing power of gold, as regards general commodities, is falling ½ per cent. per annum: a proprietor sells half his land this year for £8000 and invests the proceeds at 5 per cent. per annum; he proposes to sell the remainder three years hence and to invest the money in the same way: what income would now represent the purchasing power of the income he will have obtained when the sales are completed?

432. Seven francs are worth as much as 2 thalers, 11 thalers as much as 10 roubles, 8 roubles as much as 7 dollars, and a dollar is worth 3*s.* 11¾*d.*; a man has an equal number of the four coins: what is the least number of coins he can have in order to get an exact number of pence in exchange?

433. A man's income is £1723. 6*s.* 8*d.*: he exchanges it in Paris from English to French money: when French money is at par, 1 franc = 9·4*d.*; but at the time of his exchange it is at a discount, being worth 2·2 per cent. less than when it is at par: find how much he gets.

434. A person enters France with 33 sovereigns, 1 half-sovereign, and 7 florins: he spends 577 francs 50 centimes: how many florins will he receive in exchange for the remainder if £1 be worth 26 francs 25 centimes?

435. A vertical glass tube is partly filled with water and oil, the height of the column of oil being ·7 of that of the water, which is 25 centimetres: if 15 per cent. of the oil be removed, how much per cent. is the height of the column diminished?

436. A rectangular room has its length, breadth, and height as 7, 6, 5 respectively: its walls are decorated at a cost of 2*s.* 6*d.* per square yard, and its ceiling finished at the same rate: the bill for the whole was £86: determine the expense of covering the floor with carpet at 7*s.* per sq. yd.

437. A fast train, travelling at the rate of 36 miles per hour, leaves Dover for London, 79 miles distant, at 8 a.m., and is shortly after followed by a slow train whose speed is 24 miles an hour: the former train meets at 8.45 a.m. the express from London, which covers 48 miles in the hour, and the slower train from Dover meets the express $12\frac{1}{2}$ minutes later: find at what time the express leaves London and the slow train Dover, supposing both trains to travel at a uniform rate between Dover and London.

438. A can copy 6 pages while B copies 5, B copies 15 while C copies 12, and C copies 4 while D copies 3: A, who can write 20 pages a day, receives a paper of 240 pages to copy, and after doing a quarter of it calls in B, C, and D to help him: when will the work be finished?

439. A merchant buys 1260 quarters of corn, one-fifth of which he sells at a gain of 5 per cent., one-third at a gain of 8 per cent., and the remainder at a gain of 12 per cent.: had he sold the whole at a gain of 10 per cent., he would have gained £23. 2*s.* more: determine the cost price per quarter.

440. If the ratio of threepenny to fourpenny pieces in a given sum, which consists entirely of those coins, was altered from 3 : 7 to 7 : 3, the sum would be diminished by £20: find the sum.

441. A man after a tour in Switzerland found that he had spent every day half as many shillings as the total number of days he had been from home; his tour cost him £57. 12*s.*: how many days did the tour occupy?

442. A person fills a glass with medicine and drinks a fourth part of it: he then fills up the glass with water and drinks one-third: again he fills up with water and drinks one-half: how much of the medicine does he drink altogether, and how much each time?

443. A waggon leaves London for Oxford at 5.30 a.m., and travels at the rate of 6 miles per hour: a coach, whose rate is to that of the waggon as 5 to 3, leaves London for Oxford some time after, overtakes the waggon at 10.30 a.m., and proceeds to its destination, where, after remaining an hour, it commences its return journey, and meets the waggon $3\frac{3}{4}$ miles from Oxford: find the distance between London and Oxford, and the time (1) when the coach leaves London, (2) when it arrives at Oxford, (3) when it meets the waggon, (4) when it reaches London, and (5) when the waggon reaches Oxford.

444. A piece of metal 5·12 inches long, 3·43 inches wide, and 1·25 inches thick is melted into a perfect cube: find the length of the edge of the cube and the difference made in the amount of surface.

445. A tradesman commences business with a certain capital: at the end of the first year he adds a sixth of his capital to it; at the end of the second year he adds to it one-seventh of his new capital; at the end of the third year one-eighth of the capital with which he began that year; and at the end of the fourth year one-sixth of his last new capital: he then finds he has in the business £2737: what did he commence with?

446. *A* leaves his estate to be divided among his three sons and two daughters in the following way: the eldest son is to have one-third of the property, the second son a sum equal to two-thirds of the eldest's share, and the third a sum equal to four-fifths of the second's; the elder daughter's share is to be equal to two-fifths that of the second and third sons together, and the younger daughter is to have the remainder: it is found by this division that the younger daughter will receive £520: find the value of the estate, and what each of the others would receive.

447. The weight of a shilling being 87 grains, find the weight of a sovereign, the diameter of a sovereign being to that of a shilling as 57 to 60, and the thickness as 21 to 22, and gold being heavier than silver in the proportion of 32 to 19.

448. Two boys, *A* and *B*, start together from the same point and walk round and round a ring fence in opposite directions till they meet exactly at the starting point, their last place of meeting before this having been 990 yards from it: *A*'s rate being to *B*'s as 5 : 3, find the distance covered by each.

449. Two persons starting from the same point, and walking in opposite directions, are 25½ miles apart in 3 hours; but if they had walked in the same direction they would have been 6 miles apart in 4 hours: find their rates of walking.

450. Three guns, *A*, *B*, and *C*, are fired simultaneously at the same target, from which they are distant $\frac{3}{5}$, 1, and 2 miles respectively: the relative velocities of the shots are as 6, 5, and 4, and the shot from *A* strikes the target 3 seconds after firing; in what time after firing will the target be struck by the shots from *B* and *C*?

451. In 5 years a father will be just four times as old as his son and in 10 years he will be just three times as old: find the age of father and of son.

452. In a certain race *A* beat *B* by 80 yards; they again ran over the same course, *A*'s rate being now to his former rate as 8 to 9 and *B*'s as 9 to 10, and *A* won by 20 yds.: find the length of course.

453. In a mixture of spirits and water half the whole plus 25 gals. was spirit and a third of the whole minus 5 gals. was water: determine the number of gals. of each.

454. A labourer was engaged for 36 days upon the condition that he should receive 2*s.* 6*d.* for every day he worked, but should pay 1*s.* 6*d.* for every day he was idle: at the end of the time he received £2. 18*s.*: how many days did he work?

455. A, B, and C enter into partnership with capitals which are as 1 : 2 : 3. In 6 months A withdraws one-half of his capital, but restores it after 3 months more: at the end of 4 months from the commencement B withdraws one-third of his capital, and 2 months later C adds twice as much as B withdrew: how should a profit of £658. 10*s*. be divided at the end of the year?

456. A person who has invested a certain sum of money for $3\frac{1}{2}$ years at $4\frac{1}{2}$ per cent. per annum finds that if he had invested it and £100 more for 3 years at 5 per cent. per annum the interest would have been £11. 5*s*. more than it is: how much did he invest?

457. Three athletes run a 3-mile race round a course of 528 yards, their rates being as 32, 30, and 22: how much further have the others to run when the winner comes in?

458. A gentleman bequeaths half of his estate to his son and one-third of it to three friends, A, B, and C, in shares in the ratio of 3 : 2 : 1; he afterwards disposes of £500, and then divides the residue between B and C in shares in the ratio of 4 : 1; if C's portion comes to £6500, how much do A and B each receive?

459. A tradesman hired a man for 48 days on condition that for every day he worked he should receive 2*s*. 3*d*., and for every day he was idle he should pay 1*s*. 2*d*. for board: how many days was he idle, supposing at the end of the time he received £3. 17*s*. 3*d*.?

460. A stone pillar, in shape a cylinder of 3 feet diameter, with the top rounded into a hemisphere, is found to weigh exactly twice as much as a sphere made of the same sort of stone and of 6 feet diameter: determine the height of the pillar.

461. The area of the floors of two rooms is the same, but the volume of one of them exceeds that of the other by 1080 cubic feet: if the length and height of the larger room are 24 ft. and 15 ft. respectively, and the width and height of the smaller room are 18 ft. and 12 ft. respectively, what is the width of the larger and the length of the smaller?

462. A train going 60 miles an hour overtakes another train 160 yds. long going 45 miles an hour in the same direction on a parallel line, and passes it in half a minute: in what time would the first train pass another train of its own length going in the opposite direction at the rate of 30 miles an hour?

463. A train is travelling from one station to another: after travelling an hour it breaks down and is delayed for an hour; it then proceeds at $\frac{3}{5}$ its former rate of speed and arrives at its destination 3 hours late; had it gone 50 miles further before breaking down, it would have arrived 1 hour 20 minutes sooner: find the rate of travelling per hour and the length of the line.

464. Two casks of 48 and 42 gallons are filled with mixtures of wine and water, the proportions being 13 : 7 and 18 : 17 in the two casks respectively: if the contents of the two casks be mixed and 20 gals. of water be added to the whole, what will be the proportion of wine to water in the mixture?

465. If a person spends £50 more than $\frac{19}{20}$ of his fixed income in a certain year, £35 less than $\frac{14}{15}$ of it in the next year, and $\frac{11}{12}$ of it in the year after that, and his savings in the three years amount to £705, what is his income?

466. If, by delaying an investment in $2\frac{1}{2}$ per cent. stock while the price falls from 90 to $89\frac{3}{8}$, the annual income to be received is increased by £6. 5*s.*, what is the sum invested?

467. A flock of 37 sheep, wethers and ewes, was sold for £89. 1*s.*, a ewe fetching only three-fifths of the price of a wether; altogether the ewes fetched £4. 11*s.* more than the wethers: how many wethers were there and what did each sell for?

468. Of two square fields one exceeds the other by 100 acres, and its side is longer than that of the other by 400 yds.: find the length of the side of each field.

469. A cistern has a supply cock and also a three-way cock which can be used either for supply or for discharge; when both are used to fill it, it fills in half an hour; when the three-way cock is used to draw off, the other fills the cistern in an hour and a half, notwithstanding the discharge going on: in what time would each fill it separately?

470. A lady, wishing to relieve a number of poor people, finds that if she gives them a shilling each she will have 3*s.* 4*d.* left, and that she has not enough by 2*s.* 4*d.* to give them 1*s.* 4*d.* each: how many are there to be relieved, and how much money has she to distribute?

471. A person has two casks with wine in each; in order to have an equal quantity of wine in each he pours out of the first, which contains the larger quantity, into the second as much as the second already contains; he next pours from the second into the first as much as the first then contains, and then pours from the first into the second as much as the second contains; each cask then contains 16 gallons: how much did each contain at first?

472. The railway fare from London to Dover (70 miles) being 14*s.* 7*d.*, and that from Paris to Boulogne (240 kilometres) being 38·16 francs, compare the cost of travelling in England and France, the French metre being equal to 3 ft. 3·37168 ins., and the English sovereign being worth 25·44 francs.

473. The diameters of four spheres are to one another as 3·75 : 5 : 6·25 : 7·5; show that the greatest of these spheres is equal in volume to the other three together.

474. The square root of 9·8696044011 is 3·14159 with a remainder equivalent to ·0000166730: find in the shortest way the next 3 decimals of the root.

475. If 7 ounces of gold 22 carats fine, $12\frac{1}{2}$ ounces 21 carats fine, and 17 ounces 19 carats fine, are melted together, what is the fineness of the compound?

476. If 60 cubic inches of lead together with 54 cubic inches of cork are equal in weight to 1538$\frac{2}{3}$ cubic inches of fir, and the weights of equal quantities of lead and fir are represented by the numbers 11·324 and ·45, what number represents the weight of an equal quantity of cork?

477. *A* and *B* enter into business together, *A* providing $\frac{3}{5}$ of the capital; at the end of the first year they have made a profit of $3\frac{1}{2}$ per cent., and at the end of the second year a profit of 5 per cent., at the end of the third year they are bankrupts and can pay only 10 shillings in the pound: if they are now worth £16663 10*s.*, how much did each contribute?

478. A contractor agrees to sink an Artesian well on the following terms: £5 per fathom for the first 30 fathoms (consisting of gravel only), £35 per fathom through the sand, £20 per fathom through the clay, and £25 per fathom through the chalk; he found the thickness of the chalk to be equal to half the depth of the well, and that the sand-beds were three times as thick as the clay; if he got £2700 for sinking the well, what was the depth of it?

479. Two arm-chairs cost as much as five common chairs, and I bought a set of three arm-chairs and fourteen common chairs for £37. 12*s.* 6*d.*: determine the cost of each kind of chair.

480. One square of carpet contains 16 more square yards than another, and its side is 2 yards longer than that of the other: determine the area of each square.

481. A rectangular room is twice as long as it is broad and the height is equal to the breadth: the volume of air contained by the room is 11664 cubic feet: find the cost of carpeting the floor, the carpet being $\frac{3}{4}$ yard wide, at 5*s.* 6*d.* per yard.

482. The pressure of compressed air varies inversely as its volume: if the pressure on the inner surface of a cylinder, fitted with a piston, be 20 lbs. on the square inch, and when the piston is forced in 2 inches the pressure becomes 30 lbs. on the square inch, what is the length of the cylinder?

483. Two men, *A* and *B*, began at the same time to run in opposite ways round a circular course, starting from the same point; from the moment at which they first met, *A* took 63 seconds to finish the round and *B* took 84 seconds; but whereas *A* ran at a uniform rate the whole way, *B* mended his pace in the ratio of 4 : 3 from the moment at which they first met: find the time it took each to run the whole course.

484. The depth of a pond at one end is twice as great as at the other; eighteen inches (in depth) are drawn off, and the deep end is then three times as deep as the shallow end: determine the original depths.

485. Mercury expands under heat at the rate of $\frac{1}{10000}$ of its volume for every degree (Fah.) of temperature: find the weight of a column of mercury 6 inches in height and $\frac{3}{4}$ of a square inch in section, at a temperature of 70°, assuming the weight of a cubic foot of water at 60° to be 1000 ounces avoirdupois, and the weight of mercury at the same temperature to be 13·6 times that of water.

486. Given that the density of oxygen is 16 times as great as that of hydrogen, that the velocity of sound in hydrogen is 4200 feet per second, and that velocities of sound in gases are to one another inversely as the square roots of the densities of the gases, determine how long the sound of a pistol shot would take to travel from end to end of a tube, 1000 feet long, filled with oxygen gas.

487. A herd of 125 cattle is sold for £2575; there were half as many oxen again as there were cows, and the oxen fetched altogether £25 more than the cows: what was the price of each ox and of each cow?

488. In a certain country the births are 1 in 40, the deaths 1 in 55, and 1 person is married in every 125: find the rate of increase or decrease of the population per cent., and the average production of each marriage.

489. Find the area of the largest circular plate that can be cut out of a square sheet of iron containing 25281 inches.

490. A man purchases two estates, for the larger of which he pays twice as much as for the other: he presently sells them, and in doing so gains 15 per cent. on the cost of the smaller and $22\frac{1}{2}$ per cent. on that of the larger estate: if he receives £6498, find what he paid.

491. Find the number of square inches in the surface of a glass shade in the shape of a cylinder with a hemispherical top, the diameter of the shade being 12 inches and the total height to the top of the dome 2 feet.

492. The height of a solid six-inch cube is diminished by pressure to $5\frac{3}{8}$ inches: supposing the lateral expansion to be uniform throughout the mass, find to three places of decimals the length of one of the edges of the new case of the solid, whose volume is supposed to remain unaltered.

493. If one part of £400 be put out at 4 per cent. per annum, and the other part at 5 per cent. per annum, the yearly income obtained is £18. 5*s.* 0*d.*: find the parts.

494. A cistern 5 feet long and 3 feet 6 inches wide, filled with water, loses ·0005 of its depth of water by evaporation, and it is found that the loss by evaporation has amounted to 2 lbs. 5 ozs. weight of water: a cubic foot of water weighing $62\frac{1}{2}$ lbs., find the depth of the cistern.

495. A man embarks his whole capital in four successive speculations, in the first of which he clears 100 per cent., and in each of the others he loses 20 per cent.: determine his percentage of loss or of gain on his original capital.

496. A crown weighing $64\frac{1}{2}$ ozs. was made up of gold adulterated with silver: it was found by the water which it displaced to contain 8·5 cubic inches of metal, and a cubic inch of gold was found to weigh 10·25 ozs. and of silver 5·75 ozs.: determine the weight of gold and of silver in the crown.

497. The owner of some $3\frac{1}{4}$ per cent. stock spends in a certain year 75 per cent. of his net income after an income-tax of 6*d.* in the £ has been deducted from the interest due to him: the next year the income-tax is reduced to 3*d.* in the £; but, the rate of interest being reduced to 3 per cent., he spends only $68\frac{3}{4}$ per cent. of his net income: he then finds his surplus to be £8. 11*s.* more than in the previous year: determine how much stock he possesses.

498. A triangular space the sides of which are in the ratio $5:5:6$ contains 588 square yards: determine the lengths of its sides.

499. A merchant made a mixture of wine at 28 shillings a gallon with brandy at 42 shillings a gallon, and found that by selling the mixture at 38 shillings a gallon he gained 15 per cent. on the price of the wine and 20 per cent. on the price of the brandy: determine in what ratio the wine and the brandy were mixed.

500. If in a field, in which the grass grows uniformly, 62 oxen consume $17\frac{1}{2}$ acres in $\frac{3}{4}$ of the time in which 15 oxen consume $5\frac{1}{4}$ acres, and 11 oxen take 3 days more to consume $3\frac{3}{4}$ acres than 40 oxen do to consume $12\frac{1}{2}$ acres, find the time in which the 62 oxen consume the $17\frac{1}{2}$ acres, the original grass on each acre being supposed to be the same.

Table I.

LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	37
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	12	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	18	21	24	28
						1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	5	8	11	14	16	19	22	25
16	2041	2068	2095	2122	2148						3	5	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	15	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	2	5	7	10	12	15	17	19	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

Table I.

LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

Table II.

ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

Table II.

ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

ANSWERS.

I. Page 2.

1. 16. **2.** 2. **3.** 14. **4.** 21. **5.** 33. **6.** 12. **7.** 2.
8. 3. **9.** 17. **10.** -2.

II. Page 4.

(*a*) **1.** 8. **2.** 6. **3.** 23. **4.** 27. **5.** 11. **6.** 53. **7.** 20. **8.** 46.

(*b*) **1.** x plus the sum of y and z.
2. x plus the difference between y and z.
3. x minus the sum of y and z.
4. x minus the difference between y and z.

III. Page 5.

1. b. **2.** $-c$. **3.** $-3ab$. **4.** $x+2$.
5. $7x-16$. **6.** $20x$. **7.** $32x-13$. **8.** $3(2x-5)$.

IV. Page 6.

1. $2a+b-4c$. **2.** $4x-y+3z$. **3.** $3q+9r$. **4.** $4x^2-xy-7y^2$.

V. Page 6.

1. a^2x-ab^2x. **2.** $abxy+2cdxy$. **3.** $2a^2-3ab+b^2$.
4. $14x^2y+21xy-6xy^2-9y^2$. **5.** $12x^2-10xy-12y^2$.
6. $2m+n-4m^2n-2mn^2$. **7.** $a^3c-abc^2-a^2b^2+b^3c$.
8. $x+2x^2+xy-y-3y^2$. **9.** $8x^2+27$. **10.** $16+4x^2+x^4$.

VI. Page 10.

1. 9. **2.** 3. **3.** 6. **4.** 5. **5.** 10. **6.** $4\frac{1}{2}$.
7. 9. **8.** 10. **9.** 12. **10.** 11.

VII. Page 10.

1. $x=\frac{r+q}{y}$, $y=\frac{r+q}{x}$.

2. $x=\frac{abc}{y}$, $y=\frac{abc}{x}$.

3. $x=\frac{ab-c}{2y}$, $y=\frac{ab-c}{2x}$.

4. $x=\frac{3a+b}{y}$, $y=\frac{3a+b}{x}$.

5. $x=\frac{2b-c+d}{y}$, $y=\frac{2b-c+d}{x}$.

6. $x=\frac{pqr}{yz}$, $y=\frac{pqr}{xz}$.

7. $x=\frac{3a+b}{3y}$, $y=\frac{3a+b}{3x}$.

8. $x=\frac{7ab-2b}{5y}$, $y=\frac{7ab-2b}{5x}$.

VIII. Page 11.

1. $p-q$.

2. $2b$, or $2a$.

3. $c+d+cd$.

4. $(x-y)+(b-a)$.

5. $b+\frac{d}{c}$.

6. $2y$; $2(x+y)$.

7. $\frac{b}{a}$.

8. $(x+y)z$.

9. $c-d$.

10. $\frac{p}{q}$.

11. $x-y$.

12. $a-(b+c)$.

13. $(x+y)$ years.

14. $(p-q)$ years.

15. $(x+y)$ years.

16. $p+2$ and $p-2$.

17. $a+2$ and $a-2$.

18. $\frac{x}{y}$ years.

19. $\frac{x}{y}$ hours.

20. pq miles.

21. xy shillings.

22. $\frac{x+y}{x-y}$.

23. $\frac{n}{m}$ weeks.

24. $\frac{y}{x}$ pence.

25. $(x-y+z)$ marbles.

26. $36y$ inches, $8x$ furlongs, $\frac{3x}{12}$ shillings, $(240x+12y+z)$ pence.

27. $\frac{20a+b}{2}$ things.

28. $88x$ yards.

29. $\frac{30q}{r}$ pounds.

30. $\frac{20y}{x}$ fowls.

IX. Page 14.

1. 8.

2. £120, £30.

3. 24 inches, 8 inches.

4. £22. 10*s*., £7. 10*s*.

5. 17, 8.

6. 32, 8 years.

7. £24, £32.

8. 6.

9. 3 half-crowns and 5 florins.

10. 7 half-sovereigns, 10 half-crowns.

11. 26 apples, 13 pears.

12. 7 lbs. at 1*s*. 2*d*., 9 lbs. at 1*s*. 3*d*.

X. Page 17.

1. $x=1$, $y=1$.

2. $x=4$, $y=7$.

3. $x=3\frac{1}{2}$, $y=4$.

4. $x=12$, $y=5$.

5. $x=10$, $y=2$.

6. $x=6$, $y=2$.

7. $x=3$, $y=1$.

8. $x=1$, $y=2$.

9. $x=7$, $y=10$.

10. $x=2$, $y=4$, $z=6$.

11. $x=1$, $y=7$, $z=10$.

12. $x=3$, $y=5$, $z=7$.

13. $x=10$, $y=9$, $z=8$.

14. $x=1$, $y=5$, $z=9$.

15. $x=10$, $y=20$, $z=6$.

XI. Page 19.

1. 7, 6.
2. 18, 14.
3. Sheep 10*s.*, ox £3. 10*s.*
4. Horse £15, cow £7.
5. 37, 18.
6. 85, 20.
7. 53, 37.
8. 72.
9. Boys 4*s.*, men 9*s.*
10. Horse £12, cow £5.

XII. Page 21.

1. 2^6.
2. 2, 3^2, 7.
3. 2^2, 3, 7.
4. 2^2, 3^2, 7.
5. 2^2, 3^3.
6. 2, 3, 5, 7.
7. 2^4, 3^3.
8. 2^4, 3^2, 13.
9. 2^5, 37.
10. 2^2, 3, 5, 19.

XIII. Page 22.

1. 25.
2. 8.
3. 8.
4. 7.
5. 12.
6. 4
7. 7.
8. 12.
9. 7.
10. 2.

XIV. Page 24.

1. 3.
2. 87.
3. 29.
4. 119.
5. 73.
6. 2.
7. 217.
8. 13.
9. 74.
10. 63.
11. 121.
12. $5\frac{3}{4}$*d.*
13. 13 plots, 4 ac.
14. 18 ft.
15. 23 lbs.
16. 31 grs.

XV. Page 26.

(*a*) **1.** 81, 125, 1296, 1024, 2401.
2. 1728, 2197, 3375, 1000000, 400.
3. 11, 16, 25, 64, 35, 27.
4. 5, 6, 8, 12, 11.
5. 4500, 2700000, 35, 100.

(*b*) **1.** 525, 225, 0, 127, 9.
2. 15, 90, 931.
3. 105, 35, 2, 4, $2\frac{118}{441}$.

(*c*) $8 \times 10^4 + 5 \times 10^3 + 7 \times 10^2 + 6 \times 10 + 3$;
$3 \times 10^5 + 7 \times 10^4 + 5 \times 10^3 + 6 \times 10^2 + 8 \times 10 + 3$; $1 \times 10^6 + 5 \times 10^5$.

XVI. Page 28.

1. 144 yards.
2. £37. 10*s.* 0*d.*
3. 432 yards.
4. £990.

XVII. Page 30.

1. $4x^2 + 12xy + 9y^2$, $25x^2 + 80xy + 64y^2$, $81x^2 + 54xy + 9y^2$.
2. $4x^2 - 12xy + 9y^2$, $25x^2 - 80xy + 64y^2$, $81x^2 - 54xy + 9y^2$.
3. $4a^2 - 9b^2$, $9x^2 - 25y^2$.
4. $9x^2 - 16y^2$, $4x^4 - 16y^4$.
5. $49x^2 - 70xy + 25y^2$, $49x^2 + 70xy + 25y^2$, $36x^4 - 36x^2y^2 + 9y^4$.
6. 2401, 6084, 43681, 105625.
7. 88804, 241081, 345744, 994009.
8. 1440, 2160, 4600, 169800.

XVIII. Page 32.

1. 168.
2. 102.
3. 120.
4. 825.
5. 120.
6. 1368.
7. 7650.
8. 770.
9. 756.
10. 1560.
11. 1680.
12. 11424.
13. 1512 inches.
14. 315.
15. 5040 persons.
16. 544.
17. 3960 miles.
18. 84 seconds hence.

XIX. Page 34.

(*b*) 35, 16, 51, 4, 36, 28.

XX. Page 38.

1. $\frac{3}{5}$. **2.** $\frac{7}{13}$. **3.** $\frac{3}{4}$. **4.** $\frac{7}{16}$. **5.** $\frac{15}{29}$. **6.** $\frac{13}{37}$.
7. $\frac{19}{56}$. **8.** $\frac{9}{10}$. **9.** $\frac{8}{15}$. **10.** $\frac{107}{117}$. **11.** $\frac{5}{9}$. **12.** $\frac{13}{19}$.
13. $\frac{47}{57}$. **14.** $\frac{13}{21}$. **15.** $\frac{68}{123}$. **16.** $\frac{3}{7}$. **17.** $\frac{13}{48}$. **18.** $\frac{7}{8}$.
19. $\frac{a}{b^2}$. **20.** $\frac{y^2}{2x}$. **21.** $\frac{3n^2}{4m}$. **22.** $\frac{3q}{4p^2}$. **23.** bc^2. **24.** x^2y.

XXI. Page 39.

1. $\frac{73}{8}$. **2.** $\frac{167}{6}$. **3.** $\frac{391}{12}$. **4.** $\frac{431}{24}$. **5.** $\frac{541}{25}$.
6. $\frac{451}{15}$. **7.** $\frac{839}{20}$. **8.** $\frac{8531}{18}$. **9.** $\frac{7420}{19}$. **10.** $\frac{15406}{27}$.

XXII. Page 39.

1. $2\frac{2}{5}$. **2.** $2\frac{1}{8}$. **3.** $2\frac{7}{9}$. **4.** $2\frac{7}{16}$. **5.** $5\frac{4}{15}$.
6. $14\frac{3}{7}$. **7.** $21\frac{1}{3}$. **8.** $47\frac{13}{16}$. **9.** 108. **10.** $41\frac{12}{19}$.

XXIII. Page 40.

1. $\frac{105}{168}$, $\frac{98}{168}$, $\frac{108}{168}$, $\frac{150}{168}$. **2.** 10, 84, 154, 195, 18, 98.
3. $\frac{1}{5}$, $\frac{1}{9}$, $\frac{3}{26}$, $\frac{ab}{c}$. **4.** (1) $\frac{2}{7}$; (2) $\frac{2}{9}$.
5. (1) $1\frac{1}{8}$; (2) $12\frac{1}{2}$.

XXIV. Page 41.

1. $\frac{1}{3}$, $\frac{1}{7}$, $\frac{57}{121}$, $\frac{12}{5}$, $\frac{1}{7}$, $\frac{36x}{y}$, $\frac{112c}{d}$. **2.** $\frac{x-y-z}{x}$.
3. 3*s*. 9*d*. **4.** 15*s*.

XXV. Page 43.

1. $\frac{3}{8}$, $\frac{2}{3}$, $\frac{5}{6}$. **2.** $\frac{15}{32}$, $\frac{11}{21}$, $\frac{5}{8}$. **3.** $\frac{4}{5}$, $\frac{76}{85}$, $\frac{16}{17}$.
4. $\frac{7}{9}$, $\frac{80}{99}$, $\frac{17}{18}$. **5.** $\frac{12}{17}$, $\frac{43}{51}$, $\frac{129}{136}$, **6.** $\frac{11}{13}$, $\frac{35}{39}$, $\frac{95}{104}$.
7. $\frac{3}{8}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{6}{7}$. **8.** $\frac{2}{5}$, $\frac{7}{15}$, $\frac{11}{21}$, $\frac{2}{3}$.

XXVI. Page 44.

(*a*) **1.** 2 : 3. **2.** 11 : 28. **3.** 17 : 18. **4.** 7 : 16. **5.** 7 : 8.
6. 17 : 18. **7.** $\frac{2}{3} : \frac{1}{2}$. **8.** Equal. **9.** $\frac{4}{5} : 2\frac{1}{2}$. **10.** $1\frac{1}{2} : \frac{5}{12}$.
(*b*) **1.** $\frac{11}{30}$, $\frac{1}{30}$. **2.** $1\frac{26}{63}$, $\frac{19}{63}$. **3.** $1\frac{37}{60}$, $\frac{13}{60}$.
4. $\frac{8}{9}$, $\frac{2}{45}$. **5.** $\frac{cf+de}{df}$, $\frac{cf-de}{df}$. **6.** $\frac{qx+ry}{qy}$, $\frac{qx-ry}{qy}$.
7. $\frac{10ad+12bc}{15bd}$, $\frac{10ad-12bc}{15bd}$. **8.** $1\frac{8}{63}x$, $-\frac{x}{63}$.

(*c*) **1.** $\frac{7}{48}$, $\frac{7x}{15}$. **2.** $\frac{2}{21}$. **3.** $\frac{11}{35}$. **4.** £$\frac{22}{63}x$.
5. 20. **6.** $\frac{23}{54}$. **7.** $\frac{19}{120}x$.

XXVII. Page 46.

(*a*) **1.** $\frac{2}{5}$. **2.** $\frac{9}{32}$. **3.** $\frac{3}{20}$. **4.** $1\frac{1}{4}$.
(*b*) **1.** 7*s*. 9*d*. **2.** £5. 10*s*. 5*d*. **3.** £5. 10*s*. 0*d*. **4.** 3 tons 1 cwt. 2 qrs.
5. £$\frac{mx}{p}$. **6.** £y. **7.** x lbs. **8.** xy ounces.
(*c*) **1.** 8 guineas. **2.** 10 cwt. **3.** $4\frac{1}{2}$ cwt. **4.** 8 tons.

XXVIII. Page 46.

(*a*) **1.** $\frac{11}{12}$. **2.** $\frac{47}{48}$. **3.** $3\frac{27}{40}$. **4.** $3\frac{29}{30}$.
5. $3\frac{37}{96}$. **6.** $\frac{49}{72}$. **7.** $\frac{1}{2}$. **8.** $2\frac{13}{55}$.
9. $\frac{3}{64}$. **10.** $\frac{9}{25}$. **11.** $1\frac{5}{12}$. **12.** $\frac{1}{3}$.
(*b*) **1.** £1. **2.** 10*s*. **3.** £3. **4.** 7*s*. $4\frac{1}{2}$*d*.
5. 3 ro. 10 po. **6.** 66 lbs. 2 ozs. $7\frac{1}{9}$ drs. **7.** £1.
8. £2. **9.** 4*s*. 11*d*. **10.** £1.

XXIX. Page 48.

1. $\frac{1}{880}$. **2.** $\frac{1}{924}$. **3.** $\frac{3}{7040}$. **4.** $\frac{421}{21120}$.
5. $\frac{7}{8}$. **6.** $\frac{83}{2240}$. **7.** 2. **8.** $\frac{10}{11}$.
9. $\frac{9}{56}$. **10.** $\frac{3}{32}$. **11.** $\frac{1}{3}$. **12.** $\frac{1}{160}$. **13.** $\frac{7}{11}$.
14. $\frac{x}{220y}$. **15.** $\frac{100x}{y}$.

XXX. Page 50.

1. 112. **2.** 3025. **3.** 54 ft. **4.** 180. **5.** 78. **6.** £330.
7. £9. 12*s*. 3*d*. **8.** $(33\frac{1}{3}x+y)$ decimetres.
9. 12 tons 3 cwt. 8 lbs. **10.** $7\times10^4+4\times10^3+3\times10^2+5\times10+9$.
11. 283. **12.** —. **13.** —. **14.** $\frac{9}{17}$. **15.** $\frac{34}{81}$. **16.** —.
17. (*a*) $\frac{(r+q)b}{y}$; (*b*) 197. **18.** $\frac{240x}{y}$. **19.** $b(3x-y)$ shillings.
20. $10x$ chains. **21.** $\frac{5}{32}$ of £a. **22.** 42. **23.** 24. **24.** —.
25. 72, 9. **26.** $x=12$, $y=6$. **27.** $\frac{7}{8}$. **28.** Fowl 3*s*., goose 7*s*.

XXXI. Page 53.

(*a*) **1.** $1\frac{11}{16}$ sq. ft. **2.** $52\frac{8}{9}$ sq. yds. **3.** $55\frac{1}{8}$ sq. ch. **4.** $19\frac{1}{8}$ sq. ft.
(*b*) **1.** $1\frac{6}{7}$. **2.** $2\frac{170}{231}$. **3.** $32\frac{7}{12}$. **4.** $3\frac{13}{40}$. **5.** 1.
6. $4\frac{2}{9}$. **7.** $50\frac{5}{9}$. **8.** $86\frac{2}{3}$. **9.** $5\frac{79}{90}$. **10.** $7\frac{61}{125}$.

XXXII. Page 55.

1. £329. **2.** 21 cwt. **3.** £12. 10*s*. **4.** 7 yds. 1 ft.

5. 49 miles per hour. **6.** 12 years. **7.** 24 years. **8.** £150.

9. £13. 2*s*. 6*d*. **10.** $\frac{zx}{y}$ acres. **11.** $1267\frac{1}{2}$ ft. **12.** 3 ft. 4 ins.

13. $1\frac{11}{25}$ days. **14.** 24 days. **15.** 15 men. **16.** 108 men.

17. $78\frac{1}{5}$ acres. **18.** £2. 5*s*. 9*d*. **19.** 1*s*. $7\frac{1}{4}$*d*. **20.** £9. 6*s*. $4\frac{1}{2}$*d*.

XXXIII. Page 59.

1. $1\frac{19}{30}$. **2.** $4\frac{145}{384}$. **3.** 4. **4.** $1\frac{95}{243}$. **5.** $1\frac{5}{21}$.

6. $9\frac{9}{80}$. **7.** $1\frac{1}{3}$. **8.** $6\frac{2}{3}$. **9.** $\frac{13}{126}$. **10.** 2.

11. $2\frac{62}{77}$. **12.** $6\frac{88}{105}$. **13.** $\frac{179}{216}$. **14.** $7\frac{1}{4}$. **15.** $4\frac{17}{120}$.

16. $2\frac{3}{4}$. **17.** $\frac{245}{264}$. **18.** $1\frac{41}{51}$. **19.** $4\frac{17}{28}$. **20.** $36\frac{2}{5}$.

21. $585\frac{45}{64}$. **22.** £900. **23.** 78 tons 15 cwt. **24.** *A* £10, *B* £15.

25. *B* £160. 11*s*., *C* £414. 1*s*. **26.** $\frac{5}{4}$ of x. **27.** $2\frac{1}{2}$ ft.

28. $16\frac{1}{3}$ ft. **29.** 10*s*. **30.** $1\frac{3}{5}$.

31. £13. 13*s*. $1\frac{1}{2}$*d*. **32.** 52 tons 16 cwt. 2 qrs. 7 lbs.

XXXIV. Page 62.

1. (*a*) $6\frac{29}{72}$, (*b*) $7\frac{37}{45}$, (*c*) $3\frac{19}{240}$, (*d*) $20\frac{509}{616}$.

2. $124\frac{5}{6}$. **3.** $138\frac{1}{2}$. **4.** 2*s*. 2*d*. **5.** $9\frac{1}{6}$.

XXXV. Page 65.

1. $6\frac{19}{28}$ days. **2.** $1\frac{103}{107}$ days.

3. *A* and *B*, $2\frac{8}{11}$ days ; *A* and *C*, $2\frac{11}{12}$ days ; *B* and *C*, $3\frac{3}{13}$ days.

4. The whole. **5.** $1\frac{3}{7}$ hours. **6.** $3\frac{1}{3}$ days. **7.** $3\frac{1}{9}$ days.

8. $52\frac{1}{2}$ days. **9.** 36 days. **10.** 15 days. **11.** $\frac{24}{35}$.

12. *A*, $17\frac{3}{5}$ days ; *B*, $27\frac{1}{13}$ days. **13.** 12 m. past 5. **14.** $\frac{7}{8}$ full.

15. *A*, $21\frac{3}{17}$ days ; *B*, $27\frac{9}{13}$ days ; *C*, $51\frac{3}{7}$ days.

16. $3\frac{9}{13}$ days. **17.** $5\frac{1}{4}$ days. **18.** *A*, $12\frac{4}{73}$, *B*, $125\frac{5}{7}$, *C*, $8\frac{56}{103}$ days.

19. $23\frac{1}{3}$ hours. **20.** 5 days.

XXXVI. Page 69.

1. *A*, £5. 17*s*. $6\frac{1}{4}$*d*. ; *B*, £11. 15*s*. $0\frac{1}{2}$*d*. ; *C*, £17. 12*s*. $6\frac{3}{4}$*d*.

2. Son, £1644. 7*s*. 6*d*. ; wife, £548. 2*s*. 6*d*. ; daughter, £182. 14*s*. 2*d*.

3. *A*, £2. 8*s*. ; *B*, £1. 16*s*.

4. (1) £12. 5*s*. $3\frac{9}{11}$*d*. ; (2) £6. 2*s*. $7\frac{10}{11}$*d*. ; (3) £4. 1*s*. $9\frac{3}{11}$*d*.

5. *A*, 396 ; *B*, 297 ; *C*, 462. **6.** *A*, £5. 15*s*. 6*d*. ; *B*, £1. 11*s*. 6*d*.

7. (1) 5904 ; (2) 1476 ; (3) 492. **8.** (1) £5400 ; (2) £3600 ; (3) £2400.

9. *A*, £120 ; *B*, £40 ; *C*, £20. **10.** *A*, £36 ; *B*, £12 ; *C*, £16.

11. Sum, £26. 2*s*. 6*d*. ; *A*, £10. 10*s*. ; *B*, £8. 9*s*. ; *C*, £7. 3*s*. 6*d*.

12. *A*, £768 ; *B*, £672 ; *C*, £560.

13. *A*, £1006. 4*s*. 2*d*.; *B*, £1509, 6*s*. 3*d*.; *C*, 201. 4*s*. 10.
14. *A*, £2592; *B*, £3744; *C*, £2088.
15. *A*, 34. 13*s*. 4*d*.; *B*, £86. 13*s*. 4*d*.
16. *A* and *D*, £100 each; *B*, £115; *C*, £90.
17. Ch., 2*s*. 4*d*.; woman, 7*s*.; man, 14*s*.
18. *A*, 5*s*.; *B*, 4*s*. 6*d*.; *C*, 4*s*.
19. Man, 3*s*.; woman, 1*s*. $1\frac{1}{2}$*d*.; child, $4\frac{1}{2}$*d*.
20. £359. 18*s*. 6*d*.; £539. 17*s*. 9*d*.; £599. 17*s*. 6*d*.; £759. 16*s*. 10*d*.

XXXVII. Page 76.

1. ·6, 1·6, 26·5. 2. ·09, ·29, 1·27.

3. ·008, ·087, ·875. 4. ·0007, ·0035, ·0728.

XXXVIII. Page 82.

1. ·625. 2. ·9375. 3. 7·59375. 4. ·00375.

5. ·975. 6. ·056. 7. ·001953125. 8. 5·828125.

9. 7·015. 10. ·0784. 11. 3·71875. 12. 1·3125.

13. 1·1875. 14. ·171875. 15. ·14625. 16. ·035.

17. ·1640625. 18. ·3125. 19. ·23125. 20. ·001953125.

XXXIX. Page 85.

1. 725·2251. 2. 21·2534. 3. 13874·763903. 4. 376·32275.

5. 56·943175. 6. 74·760949. 7. 27·7777775. 8. 100·1111111.

9. 1134·218 m. 10. 784·022 g. 11. 5873·411 l. 12. 278·9 fr.

13. 58·5903 Dm. 14. 6047·27 hl.

15. 378·42, 37842. 16. 594·86 hm., 59·486 hm.

XL. Page 86.

1. ·99. 2. 74·925. 3. 22·5124. 4. 16·9983.

5. 155·52. 6. 1·43856. 7. 6·399936. 8. 475·695.

9. ·09979. 10. ·0009. 11. 9·413. 12. ·98976.

13. 86·7358. 14. 1·1824. 15. 52·594 m. 16. 9·27 fr.

17. 282·587 l. 18. 153·565 g.

XLI. Page 89.

1. 1156·725 m., 1·972 m. 2. 101·41686 hg. 3. 43·81153 Kg.

4. 63·0666 Dl. 5. 11·005443 Kl. 6. 36·225 fr.

7. 68·375 fr. 8. $10000x$ cm. 9. $100y$ centimes.

10. $10xy$ fr. 11. 15·98 sq. yds. 12. 26·68 sq. m.

13. 23·04 sq. Dm. 14. 32·379. 15. 1·8019.

16. 158466·011. 17. 112·9115. 18. 6861·90045. 19. ·056658.

20. ·005628. 21. ·003. 22. ·01653377. 23. ·000080223.

24. ·251. 25. 2400·6.

XLII. Page 93.

1. 7·25 m. **2.** 39·672 g. **3.** 45·279 l. **4.** 81·076 g.
5. 7·25 m. **6.** 46. **7.** 19·75 m. **8.** 75 steps.
9. 35. **10.** 58 lambs. **11.** 9640. **12.** 3490000.
13. 7060. **14.** 240. **15.** 7500. **16.** 24600.
17. 9600. **18.** 67500. **19.** 22500. **20.** 817000.
21. ·603. **22.** ·013. **23.** 13. **24.** 130.
25. ·00012. **26.** ·0096. **27.** 130. **28.** ·00125.
29. ·0135. **30.** 1·5. **31.** 8·06125. **32.** 6·1.

XLIII. Page 97.

1. 53·4, 53·37, 53·366, 53·3658. **2.** 44·2, 44·17, 44·167, 44·1673.
3. 3·7909. **4.** 1·6088. **5.** ·0187. **6.** 9·3777.

XLIV. Page 98.

1. ·4106861. **2.** ·3678794. **3.** 1·499999. **4.** ·2876.

XLV. Page 99.

1. To units ; error ± ·02. **2.** To units ; error ± ·025.
3. To the second place ; error ± ·002. **4.** To dm. ; error ± ·001.
5. To dg. ; error ± ·001 mg. **6.** To grams ; error ± ·01 gr.

XLVI. Page 102.

1. 9·59749. **2.** 3·70710. **3.** 5·2976. **4.** 1·16643.
5. 1·411993. **6.** 78·830.

XLVII. Page 103.

1. Two, 17 sq. m. ; error ± ·0421 sq. m.
2. Four, 11·16 sq. m. ; error ± ·0034595 sq. m.
3. Three, 25·8 sq. in. ; error ± ·02565 of a sq. in.
4. 4407·758 and 4383·812. **5.** ± ·014.

XLVIII. Page 105.

1. 84961. **2.** 8·461. **3.** ·055522.
4. ·053749. **5.** ·009841. **6.** 65·00762.

XLIX. Page 107.

1. (*a*) Two, 2·2 ; (*b*) Two, 7·7. **2.** 73·4 and 73·1 m.
3. £12. 12*s*. 2*d*. and £12. 10*s*. 3½*d*. **4.** 1*s*. 5·60*d*. and 1*s*. 4·82*d*.
5. Three, 1·475. **6.** $14{\cdot}35\dot{3}$ and 12·6529.... **7.** 133 and 112.

L. Page 110.

1. 57·6. **2.** 16·575 ft. **3.** 457 times. **4.** 99 minutes.
5. 12 seconds. **6.** $\frac{1}{1800}$. **7.** $12\frac{21}{22}$ yds. **8.** 132 ft.
9. 24. **10.** 112. **11.** 90. **12.** £72.
13. £600. **14.** 2. **15.** $\frac{m}{n}$ miles, $\frac{60n}{m}$ minutes, $\frac{nx}{m}$ hours.

LI. Page 114.

(*a*) **1.** $\cdot\dot{6}$. **2.** $\cdot\dot{8}$. **3.** $\cdot 17\dot{8}5714\dot{2}$.
4. $\cdot\dot{8}5714\dot{2}$. **5.** $\cdot\dot{4}2857\dot{1}$. **6.** $\cdot\dot{3}0769\dot{2}$.
7. $\cdot 4\dot{6}$. **8.** $\cdot 7\dot{8}5714\dot{2}$. **9.** $\cdot 12\dot{1}4285\dot{7}$.
10. $\cdot 2\dot{6}1538\dot{4}$. **11.** $\cdot\dot{0}58823529411764\dot{7}$. **12.** $\cdot\dot{1}5789473684210526\dot{3}$.
(*b*) **13.** $\cdot 3\dot{7}\dot{5}$. **14.** $\cdot 20\dot{3}$. **15.** $\cdot 82886\dot{9}0476\dot{1}$.
16. $\cdot\dot{0}94\dot{6}$. **17.** $\cdot 0011\dot{3}\dot{6}$. **18.** $\cdot 8\dot{3}$.
19. $\cdot 501041\dot{6}$. **20.** $\cdot 01988\dot{6}\dot{3}$.

LII. Page 117.

1. $\frac{1}{3}$. **2.** $\frac{6}{11}$. **3.** $\frac{4}{37}$. **4.** $2\frac{17}{99}$.
5. $7\frac{56}{111}$. **6.** $\frac{1}{88}$. **7.** $\frac{7}{55}$. **8.** $\frac{3}{275}$.
9. $\frac{17}{72}$. **10.** $3\frac{7}{108}$. **11.** $\frac{1}{375}$. **12.** $\frac{13}{396}$.
13. $5\frac{1}{55}$. **14.** $3\frac{4}{333}$. **15.** $2\frac{1}{2457}$. **16.** $\frac{17463}{55550}$.

LIII. Page 120.

1. $\cdot 8\dot{4}3\dot{0}$. **2.** $\cdot 69\dot{3}41068\dot{8}$. **3.** $\cdot 52\dot{4}8639\dot{5}$.
4. $\cdot 0\dot{5}8704\dot{1}$. **5.** $4\cdot 08\dot{5}9949238167\dot{0}$. **6.** $\cdot 8\dot{3}71\dot{9}$.
7. 1. **8.** $\frac{2}{7}$ or $\cdot\dot{2}8571\dot{4}$. **9.** $\frac{5}{27}$ or $\cdot\dot{1}8\dot{5}$.
10. $\frac{2}{39}$ or $\cdot\dot{0}5128\dot{2}$. **11.** $\frac{5}{33}$ or $\cdot\dot{1}\dot{5}$. **12.** $\frac{7}{8}$ or ·875.
13. $5\frac{9}{11}$ or $5\cdot\dot{8}\dot{1}$. **14.** $\frac{26}{495}$ or $\cdot 0\dot{5}\dot{2}$. **15.** $61\frac{1}{9}$ or $61\cdot\dot{1}$.
16. $2\frac{323}{500}$ or 2·646. **17.** ·185. **18.** ·28571.
19. ·05128. **20.** ·1515. **21.** 1·33884.
22. ·625. **23.** 1·04907. **24.** 151·397.

LIV. Page 121.

1. 1300. **2.** 65. **3.** $\frac{1}{160}$. **4.** $\frac{3}{7}$.
5. $\frac{1}{16}$. **6.** $\frac{3}{190}$. **7.** 40·2175. **8.** 4·8.
9. $\frac{107}{199800}$, $\cdot 000\dot{5}3\dot{5}$. **10.** 54·375, 45·625. **11.** 153 times, ·011 pint.
12. 35, ·27 inch. **13.** £234. 2*s*. 6*d*. **14.** ·975.
15. 6·232. **16.** ·2205. **17.** 6·7, 6·5, 3·1, 1·9.
18. 25 : 27. **19.** 133 : 103. **20.** ·0441.

21. ·421875. **22.** $7{\cdot}34\times10^{-2}$, $6{\cdot}45\times10^{-4}$.

23. 105. **24.** $\frac{adf}{bdf}$, $\frac{cbf}{bdf}$, $\frac{ebd}{bdf}$.

25. $\frac{2x^2+3x+a}{5x}$. **26.** $17\frac{1}{7}$. **27.** 5.

LV. Page 127.

1. 2·5563, 4·5563, $\bar{2}{\cdot}5563$, $\bar{4}{\cdot}5563$, $\bar{6}{\cdot}5563$.

2. 1·5441, 3·5441, $\bar{3}{\cdot}5441$, $\bar{5}{\cdot}5441$. **3.** 1·8573, 4·8573, $\bar{3}{\cdot}8573$, $\bar{6}{\cdot}8573$.

4. (*a*) 1·3222, (*b*) 1·1461, (*c*) 1·8865, (*d*) 1·5185, (*e*) ·5643, (*f*) ·5441, (*g*) ·1963.

LVI. Page 129.

1. 2·4330, ·4330, $\bar{1}{\cdot}4330$, $\bar{3}{\cdot}4330$. **2.** 3·4675, 1·4675, $\bar{1}{\cdot}4675$, $\bar{4}{\cdot}4675$.

3. 1·4771, ·4771, $\bar{2}{\cdot}4771$, $\bar{6}{\cdot}4771$.

LVII. Page 131.

1. 1·833, 18·33, 183·3, ·1833. **2.** 1·883, 18·83, 188·3, ·01883.

3. 1·973, 19·73, 1973, ·001973.

LVIII. Page 133.

1. 127·2. **2.** 177·3. **3.** 2863 sq. m.

4. 9170 lbs. **5.** 2·237 cub. m. **6.** 2·82 sq. m.

LIX. Page 134.

1. 22·91. **2.** ·008476. **3.** 49·22 m. **4.** 2·233.

LX. Page 137.

1. $\bar{6}{\cdot}7647$, $\bar{7}{\cdot}0196$, $\overline{15}{\cdot}4575$. **2.** $\bar{4}{\cdot}7351$, $\bar{6}{\cdot}5585$, $\bar{9}{\cdot}2936$.

3. $\bar{1}{\cdot}4756$, $\bar{1}{\cdot}5805$, $\bar{1}{\cdot}6504$. **4.** $\bar{1}{\cdot}6527$, $\bar{1}{\cdot}7395$, $\bar{1}{\cdot}7916$, $\bar{1}{\cdot}8264$.

LXI. Page 137.

1. 51·86. **2.** 70·55. **3.** 16·35. **4.** 19·66. **5.** 6·252.

6. ·2319. **7.** ·1945. **8.** ·4083. **9.** ·2178.

10. 459·9. **11.** 61·91. **12.** 3·406. **13.** 6·668.

14. 1·826. **15.** 4·283 in. **16.** 45·82 c. dm. **17.** 9·752 in.

LXII. Page 140.

1. 18*s.* 9*d.*, 1 ton 8 lbs. 6 ozs. $6\frac{2}{5}$ drs., 4919·86 yds.

2. 2 tons 3 qrs. 15 lbs. 12 ozs. **3.** £3. 12*s.* $2\frac{1}{4}$*d.*, 3·3*d.*

4. $1\frac{1}{2}$*d.*, £142. 6*s.* 3*d.* **5.** 19*s.* 9*d.*

6. 1 ton 18 cwt. 3 qrs. 19·0765 lbs. **7.** 3 qrs. 9 lbs. 6 ozs.

8. 9·16 ozs. **9.** 125 tons. **10.** 5 tons 16 cwt. 1 qr. $7\frac{1}{2}$ lbs.

LXIII. Page 141.

1. $\cdot 8\dot{3}$. **2.** $\cdot 00208\dot{3}$. **3.** ·6625. **4.** ·84375.
5. ·565. **6.** ·89375. **7.** 4·2703125. **8.** ·409296875.

LXIV. Page 146.

1. $\left.\begin{array}{l}\text{£}2\cdot859,\\ \text{£}2\cdot859375;\end{array}\right\}$ $\left.\begin{array}{l}\text{£}3\cdot285,\\ \text{£}3\cdot28541\dot{6};\end{array}\right\}$ $\begin{array}{l}\text{£}6\cdot688,\\ \text{£}6\cdot688541\dot{6}.\end{array}$

2. $\left.\begin{array}{l}\text{£}5\cdot857,\\ \text{£}5\cdot857291\dot{6};\end{array}\right\}$ $\left.\begin{array}{l}\text{£}7\cdot193,\\ \text{£}7\cdot19375;\end{array}\right\}$ $\begin{array}{l}\text{£}8\cdot930,\\ \text{£}8\cdot930208\dot{3}.\end{array}$

3. $\left.\begin{array}{l}\text{£}1\cdot831,\\ \text{£}1\cdot83125;\end{array}\right\}$ $\left.\begin{array}{l}\text{£}3\cdot128,\\ \text{£}3\cdot128125;\end{array}\right\}$ $\begin{array}{l}\text{£}10\cdot896,\\ \text{£}10\cdot896875.\end{array}$

4. $\left.\begin{array}{l}\text{£}3\cdot976,\\ \text{£}3\cdot976041\dot{6};\end{array}\right\}$ $\left.\begin{array}{l}\text{£}9\cdot886,\\ \text{£}9\cdot886458\dot{3};\end{array}\right\}$ $\begin{array}{l}\text{£}6\cdot642,\\ \text{£}6\cdot642708\dot{3}.\end{array}$

5. 15*s*. $4\frac{3}{4}$*d*., 1*s*. $4\frac{1}{2}$*d*., 2*s*. 11*d*., 17*s*. $2\frac{1}{4}$*d*.
6. £45. 16*s*. 7*d*. **7.** £57. 15*s*. 10*d*. **8.** £1. 17*s*. 4*d*.

LXV. Page 148.

1. £52. 4*s*. 9*d*. **2.** £105. 17*s*. 5*d*. **3.** £41. 11*s*. 8*d*.
4. £210. 6*s*. 5*d*. **5.** £308. 4*s*. **6.** £62. 8*s*. 6*d*.
7. £281. 4*s*. 6*d*. **8.** £35. 10*s*. 9*d*. **9.** £51. 2*s*. 2*d*.
10. £465. 11*s*. 7*d*. **11.** £2493. 4*s*. 3*d*. **12.** £50. 12*s*. 6*d*.
13. £512. 19*s*. 9*d*. **14.** £19. 9*s*. 7*d*. **15.** £165. 0*s*. 3*d*.
16. £2749. 9*s*. **17.** £150. 14*s*. 3*d*. **18.** £57. 1*s*. 2*d*.
19. £159. 7*s*. 9*d*. **20.** £89. 9*s*. 2*d*. **21.** £103. 10*s*. 11*d*.
22. £88. 4*s*. **23.** £19. 11*s*. 9*d*. **24.** £9532. 18*s*. 5*d*.
25. £6539. 6*s*. 11*d*. **26.** £29. 12*s*. 10*d*. **27.** £45. 19*s*. 6*d*.
28. £113. 17*s*. 4*d*. **29.** £227. 2*s*. 2*d*. **30.** £44. 13*s*. 6*d*.

LXVI. Page 153.

1. £661. 10*s*. **2.** £631. 10*s*. **3.** £3791. 11*s*. $3\frac{1}{4}$*d*.
4. £8. 2*s*. 6*d*. **5.** £3765. 10*s*. **6.** £2184. 15*s*.
7. £722. 13*s*. 4*d*. **8.** £2500. **9.** £2560.
10. 7*s*. 10*d*. **11.** 5*s*. **12.** 13*s*. 6*d*. **13.** £86. 1*s*. $10\frac{1}{2}$*d*.
14. 3*s*. 5*d*. **15.** 4*s*. $9\frac{3}{5}$*d*.; £252. **16.** £786. 2*s*. 6*d*.
17. £1800. **18.** £321. 6*s*. **19.** £451. 10*s*.
20. £500, £100, £6. 10*s*., £10.
21. *A*, £81. 11*s*. 3*d*.; *B*, £156. 7*s*. 6*d*.; *C*, £174. 7*s*. 6*d*.
22. £412. **23.** *A*, £$828\frac{4}{7}$; *B*, £$787\frac{1}{7}$; *C*, £$662\frac{6}{7}$; *D*, £$621\frac{3}{7}$.

LXVII. Page 156.

1. $16:27$. **2.** $ae:bf$. **3.** $b:d$.
4. $\left.\begin{array}{l}4:9,\\ 8:27;\end{array}\right\}$ $\left.\begin{array}{l}25:49,\\ 125:343;\end{array}\right\}$ $\left.\begin{array}{l}c^2:d^2,\\ c^3:d^3;\end{array}\right\}$ $\left.\begin{array}{l}x^2y^2:z^2,\\ x^3y^3:z^3;\end{array}\right\}$ $\begin{array}{l}a^2b^2:c^2d^2,\\ a^3b^3:c^3d^3.\end{array}$

LXVIII. Page 162.

1. 147. **2.** 162. **3.** 15. **4.** 18.

5. $\frac{py}{q}$. **6.** $\frac{ny}{m}$. **7.** $\frac{my}{n}$. **8.** $\frac{4y}{3}$.

9. $a:5=b:9$; $9:b=5:a$; $9:5=b:a$. **10.** 8.

11. 12. **12.** 14. **13.** $\sqrt{xy}$. **14.** 16. **15.** $2\sqrt{ab}$.

16. 20. **17.** 49. **18.** 9. **19.** $\frac{3}{32}$. **20.** $\frac{25}{18}$.

21. $\frac{b^2}{a}$. **22.** $\frac{5}{12}$. **23.** 168. **24.** £420. **25.** $21\frac{1}{5}$ yards.

26. $112\frac{8}{11}$ yards. **27.** 3 points. **28.** 18 yards.

29. $30\frac{28}{45}$ yards. **30.** $\frac{y(d-c)}{d(y-x)}$.

LXIX. Page 168.

1. £3. 11*s*. 6*d*. **2.** $1\frac{3}{4}$ yds. **3.** ·0625 ch.

4. 60 miles. **5.** £350. **6.** 60 cwt.

7. 2 lbs. 3 ozs. **8.** 4 lbs. **9.** $5\frac{5}{7}$ days.

10. $25\frac{5}{19}$ days. **11.** $9\frac{3}{4}$ weeks. **12.** 35 days.

13. 4 wks. 2 days. **14.** 2800 men. **15.** 1000 men.

16. 160 men. **17.** 15 days. **18.** 12 weeks.

19. 48 men. **20.** 452·389 sq. yds. (nearly). **21.** 25 cub. ft.

22. 5·4 cub. ft. **23.** 480 sq. ch. **24.** 80 gals.

25. $21\frac{1}{3}$ lbs. **26.** 216 bush. **27.** $\frac{1}{69696}$.

LXX. Page 172.

1. £2. 13*s*. $7\frac{1}{2}$*d*. **2.** £8. **3.** $57\frac{33}{35}$ qrs.

4. 15 persons. **5.** 11 hours. **6.** $13\frac{1}{2}$ inches.

7. $13\frac{1}{2}$ days. **8.** 18 men. **9.** 1320 yds.

10. $10\frac{145}{512}$ days. **11.** $222\frac{3}{4}$ days. **12.** 12 hours.

13. 32 days. **14.** £50. **15.** 8 days.

16. 5 months. **17.** £50. 10*s*. **18.** 11 days.

19. 20 weeks. **20.** 1000 miles. **21.** $3\frac{3}{4}$ days.

22. £7. 11*s*. 3*d*. **23.** £214. 15*s*. $5\frac{5}{11}$*d*. **24.** 27 months.

25. £53. 15*s*. $2\frac{2}{5}$*d*. **26.** $7\frac{1}{9}$ hours. **27.** $73\frac{3}{4}$ days (nearly).

28. $7\frac{1}{11}$ ozs. **29.** $9\frac{3}{5}$. ozs. **30.** $12\frac{1}{2}$ ozs.

31. $19\frac{1}{4}$ ozs. **32.** 1 : 13·6. **33.** 31·39... stones.

34. 16 lbs. **35.** 14·75....

LXXI. Page 182.

1. Boys, 156; girls, 104. **2.** £23. 6*s*. 8*d*.; £46. 13*s*. 4*d*.; £70.

3. £1872; £4992; £5148. **4.** *A*, £16. 3*s*.; *B*, £24. 4*s*. 6*d*.; C, £40. 7*s*. 6*d*.

5. *A*, £19. 5*s.*; *B*, £27. 2*s.* 6*d.*; *C*, £37. 12*s.* 6*d.*
6. £4285. 14*s.* $3\frac{3}{7}$*d.*; £2857. 2*s.* $10\frac{2}{7}$*d.*; £2142. 17*s.* $1\frac{5}{7}$*d.*; £1714. 5*s.* $8\frac{4}{7}$*d.*
7. £2250; £1500; £1125.   8. Boys, £1050; girls, £675.
9. 192, 150.   10. 12 half-crs., 24 pence, 36 sovs., 96 shillings.
11. *A*, £48; *B*, £72; *C*, £180.   12. *C*, £630; *B*, £600; *A*, £400.
13. *A*, £424. 4*s.*; *B*, £707; *C*, £777. 14*s.*
14. *A*, £852. 4*s.*; *B*, £1278. 6*s.*; *C*, £1597. 17*s.* 6*d.*
15. *A*, £135. 15*s.*; *B*, £108. 12*s.*; *C*, £81. 9*s.*; *D*, £90. 10*s.*
16. *A*, £131. 5*s.*; *B*, £350; *C*, £787. 10*s.*; *D*, £210.
17. £3267; £2662; £2057.   18. *A*, £780; *B*, £1040.
19. *A*, £336; *B*, £420; *C*, £210.   20. *A*, £240; *B*, £225; *C*, £200.
21. £1732. 10*s.*; £5775; £2310.   22. *A*, £120; *B*, £90; *C*, £192.
23. *A*, £21. 16*s.* $4\frac{4}{11}$*d.*; *B*, £13. 12*s.* $8\frac{8}{11}$*d.*; *C*, £4. 10*s.* $10\frac{10}{11}$*d.*
24. £22. 10*s.*, £40, £58. 10*s.*
25. *A*, £2. 19*s.* 7*d.*; *B*, £11. 18*s.* 4*d.*; *C*, £10. 14*s.* 6*d.*
26. *A*, £168; *B*, £280.   27. *A*, £50; *B*, £60.
28. *A*, £90; *B*, £75; *C*, £105.   29. *A*, £245; *B*, £175; *C*, £210.
30. *A*'s cap., £4000; *B*'s, £2500; *A*'s profit, £$1523\frac{1}{13}$; *B*'s, £$951\frac{12}{13}$.
31. *A*, £23. 5*s.* 9*d.*; *B*, £30. 14*s.* 3*d.*
32. £1111. 2*s.* $2\frac{2}{3}$*d.*; £2222. 4*s.* $5\frac{1}{3}$*d.*; £1666. 13*s.* 4*d.*; £5000.
33. £750.   34. *A*, 240; *B*, 360; *C*, 300; *D*, 400.
35. *A*, 800; *B*, 500; *C*, 300.
36. 48 half-sovs.; 96 half-crowns; 176 threepences.
37. 69 half-sovs.; 115 half-crowns; 161 sixpences.
38. *A*, £384; *B*, £384; *C*, £288.   39. 9 months.
40. { *A*'s capital, £1900; *B*'s, £1500; *C*'s, £1200; *A*'s profits, £142. 10*s.*;
B's, £112. 10*s.*; *C*'s, £90.
41. 2 lbs. 4 ozs. 1 dwt. 14·4 grs. of copper; 7 ozs. 18 dwts. 9·6 grs. of zinc.
42. 5 dwts. $14\frac{12934}{70399}$ grs.   43. 3 dwts. $8\frac{8}{11}$ grs.

LXXII. Page 188.

1.—12.
13. 1 sq. in.   14. ·9 sq. in.   15. 1·28 sq. in.
16. 1·8 sq. in.   17. 1·39 in.   18. 1·7 in.

LXXIII. Page 197.

1. £1. 1*s.* 3*d.*, £1. 10*s.*; $y=1\frac{1}{4}x$.   2. —
3. $y=2x+3$, 15 oranges, 9*d.*   4. $y=1\frac{1}{2}x+2$, 8*s.*, 11*s.*, 12*s.* 6*d.*, 15*s.* 6*d.*
5. $y=1{\cdot}82x$.   6. —.   7. 400 miles, 900 km.   8. —.   9. —.

LXXIV. Page 201.

1. $62\frac{1}{2}$, $44\frac{4}{9}$, 45, $37\frac{1}{2}$, 250, 15, 70, 575.

2. 120, $31\frac{1}{4}$, $12\frac{1}{2}$, 75, $100x$, $\frac{100}{x}$, $\frac{100z}{y}$, 440.

3. $\frac{1}{25}$, $\frac{3}{20}$, $\frac{4}{5}$, $\frac{19}{20}$, $\frac{1}{8}$, $\frac{7}{200}$. **4.** $\frac{9}{200}$, $\frac{7}{400}$, $\frac{1}{200}$, $\frac{3}{400}$, $\frac{1}{800}$, $\frac{x}{10}$;

5. £7. 10*s.*, £3. 7*s.* 6*d.*, £5. **6.** 40, $16\frac{2}{3}$, 10, $33\frac{1}{3}$.

7. $4\frac{2}{5}$, $13\frac{3}{19}$. **8.** £25, 16*s.* 9*d.*, 50 ft. **9.** 20 *g.*, 89 *m.*, £20.

LXXIV. (a). Page 202.

1. 12*s.* 6*d.*, £2. 1*s.*, £2. 7*s.* 9*d.* **2.** 13*s.* 6*d.*, £2. 10*s.*, £1. 13*s.* 9*d.*

3. £3. 7*s.* 6*d.*, £2. 1*s.* 6*d.*, £3. 12*s.* 6*d.* **4.** £1. 1*s.*, £14. 3*s.* 4*d.*, £11. 5*s.*

LXXV. Page 203.

1. £14. 6*s.* 10*d.* **2.** £4. 8*s.* 2*d.* **3.** £64. 7*s.* 8*d.* **4.** £10. 18*s.* 10*d.*

LXXVI. Page 206.

1. $33\frac{1}{3}$. **2.** 25. **3.** Boys, 43·13; girls, 27·09; infants, 29·76.
4. 38,896,200. **5.** $64\frac{3}{5}$. **6.** 87·5, 93·75, 84·375 respectively.
7. 90, 87·5, 85 respectively. **8.** 468, 572, 546 respectively.
9. 97·4, 87·7, 76·02. **10.** £600. **11.** £1390.
12. £7500. **13.** $91\frac{1}{5}$. **14.** 579. **15.** £1,250,000.
16. £315. **17.** ·013. **18.** 5. **19.** $9\frac{1}{11}$.
20. 500. **21.** $17\frac{4}{25}$. **22.** £1500. **23.** £1800.
24. £2700. **25.** £8,000,000. **26.** £7,358,400.

LXXVII. Page 209.

1. £8. 18*s.* $3\frac{3}{4}$*d.* **2.** £2000. **3.** £18. 2*s.* 6*d.*
4. £651. 2*s.*; £14668. 18*s.* **5.** £4488. 15*s.*

LXXVIII. Page 217.

1. 12. **2.** $10\frac{5}{7}$. **3.** $14\frac{2}{7}$. **4.** $71\frac{29}{51}$.
5. £65. 13*s.* **6.** 3*s.* $4\frac{1}{4}$*d.* **7.** £40. **8.** £1.
9. 19*s.* 2*d.* **10.** 3*s.* 2*d.* (nearly). **11.** 4 p.c. loss. **12.** $7\frac{1}{2}$ p.c. gain.
13. $62\frac{1}{2}$. **14.** $106\frac{1}{4}$. **15.** 1*s.* $1\frac{1}{2}$*d.* **16.** £95. 16*s.* 8*d.*
17. $15\frac{5}{13}$. **18.** 50. **19.** $5\frac{3}{4}$ p.c. gain. **20.** $5\frac{85}{123}$.
21. £7. **22.** $\frac{1}{2}$. **23.** 50. **24.** 7*s.* 4*d.*, 60.
25. 65 p.c. loss. **26.** 25. **27.** $12\frac{1}{2}$. **28.** $31\frac{1}{4}$.
29. 80. **30.** 200. **31.** 400. **32.** 32.
33. $3\frac{19}{27}$. **34.** $10\frac{10}{29}$. **35.** $30\frac{5}{64}$. **36.** $32\frac{1}{4}$.
37. £20. 12*s.* 6*d.*, 25 p.c. **38.** £250, 30 p.c.

39. 10*s.* 5*d.* **40.** 15*s.* **41.** £40. **42.** $9\frac{3}{5}$*d.*
43. £1. 0*s.* 10*d.*, 17*s.* $3\frac{1}{2}$*d.* **44.** 3 cwts. 2 qrs. 23 lbs. **45.** 9*d.*
46. $2\frac{6}{7}$ pence each. **47.** £1. 2*s.* 6*d.* **48.** £1. 0*s.* 10*d.*
49. £40. **50.** 18*s.* **51.** 15*s.* **52.** £2. 4*s.*
53. 8*s.* 4*d.* **54.** £66. 13*s.* 4*d.* **55.** £75.
56. Tea, 3*s.* 4*d.*; sugar, 5*d.* **57.** Tea, 2*s.* 6*d.*; sugar, 6*d.*
58. Horse, £15; cow, £10. **59.** Tea, 3*s.* 9*d.*; coffee, 1*s.* 8*d.*
60. £48. **61.** 2*d.* **62.** $12\frac{1}{2}$.

LXXIX. Page 224.

1. 3 : 1. **2.** 2 : 1. **3.** 7 : 4. **4.** 3 : 3 : 1.
5. 1 : 2 : 5 : 1. **6.** 2 lbs. at 5*d.*; 3 lbs. at 7*d.*; 5 lbs. at 12*d.*; 3 lbs. at 13*d.*
7. 2 : 13. **8.** 2 : 7. **9.** (1) 3 lbs. of 1st, 1 lb. of 2nd, 12 lbs. of 3rd; (2) 408 lbs.; (3) 45 lbs., 15 lbs., 180 lbs.
10. 28 lbs. at 5*s.*; 56 lbs. at 6*s.*; 56 lbs. at 8*s.*; 28 lbs. at 9*s.*; or 42 lbs. of each.
11. (1) 5 : 4; (2) 15 lbs. and 12 lbs.
12. 9 gals. at 12*s.*; 18 gals. at 13*s.*; 27 gals. at 14*s.*; 9 gals. at 14*s.* 6*d.*
13. $4\frac{4}{5}$. **14.** 50 lbs. of brass; 200 lbs. of pewter.
15. 44 gals. at 7*s.* 5*d.*; 16 gals. at 5*s.* 2*d.*; 34 gals. at 4*s.* 2*d.*
16. $185\frac{5}{8}$ lbs. **17.** 9*s.* 8*d.* **18.** 6*d.* and 4*d.*
19. £40, £80. **20.** $33\frac{1}{3}$ cubic feet.

LXXX. Page 231.

1. £230. **2.** £46. 16*s.* **3.** £166. 5*s.*
4. £69. 18*s.* 1*d.* **5.** £8. 8*s.* 4*d.* **6.** £44. 4*s.* 3*d.*
7. £7. 11*s.* 2*d.* **8.** £1. 5*s.* **9.** £475. 15*s.* 3*d.*
10. £495. 14*s.* 2*d.* **11.** £662. 4*s.* 11*d.* **12.** £36. 13*s.* 4*d.*
13. £1244. 10*s.* 1*d.* **14.** £23. 12*s.* **15.** £193. 4*s.* 6*d.*
16. £31. 1*s.* **17.** £13. 10*s.* 2*d.* **18.** £1. 4*s.* 11*d.*
19. £27. 10*s.* 7*d.* **20.** $3\frac{1}{2}$. **21.** 3.
22. $3\frac{1}{2}$. **23.** 12. **24.** 50.
25. $5\frac{1}{2}$. **26.** 4 yrs. **27.** 4 yrs.
28. $3\frac{1}{2}$ yrs. **29.** 6 months. **30.** £227. 10*s.*
31. £1020. **32.** £566. 13*s.* 4*d.* **33.** £2250; £37. 10*s.*

LXXXI. Page 234.

1. £1. 5*s.* **2.** £231. 15*s.* **3.** £6. 9*s.* $11\frac{1}{4}$*d.*
4. £180. **5.** 8 p.c. **6.** £1800.

LXXXII. Page 238.

1. 1*s.* 5*d.* nearly. **2.** 5*s.* nearly. **3.** £25. **4.** £858. 6*s.* 8*d.*
5. £158. 17*s.* 6*d.* **6.** 1 year. **7.** £553. 6*s.* 8*d.*; 5 p.c.

LXXXIII. Page 241.

1. $3\cdot1\dot{3}$ months.   **2.** $4\frac{1}{2}$ months.   **3.** $4\frac{5}{8}$ months.

4. 38 days.   **5.** 269 days, Dec. 26, 1906.

LXXXIV. Page 246.

1. £9. 12*s.*; £9. 12*s.* 3*d.*   **2.** £484; £484. 11*s.* 11*d.*

3. £5190; £5191. 13*s.* 7*d.*   **4.** £607. 16*s.*; £607. 15*s.* 1*d.*

5. £919. 8*s.*; £919. 12*s.* 8*d.*   **6.** £3129; £3129. 5*s.*

7. £550. 2*s.*; £550.   **8.** £324. 14*s.*; £324. 14*s.* 7*d.*

LXXXV. Page 249.

1. £2000.   **2.** £260.   **3.** £375.   **4.** 3 years.

5. 2 years.   **6.** 3 p.c.   **7.** 3 p.c.   **8.** 17·6 years.

9. 37·2 years.   **10.** £73·1, correct to tenths.   **11.** 5 p.c.; £8. 6*s.* 8*d.*

12. £500.   **13.** £96.   **14.** £2500.

LXXXVI. Page 256.

1. £11643. 15*s.*   **2.** £19872.   **3.** £5208. 6*s.* 8*d.*

4. £550; £38. 10*s.*   **5.** £135.   **6.** £92. 5*s.*

7. £6666. 13*s.* 4*d.*   **8.** £60,000.   **9.** £3000.

10. £14,000.   **11.** $95\frac{7}{8}$.   **12.** $90\frac{3}{8}$.

13. 105.   **14.** $92\frac{7}{8}$.   **15.** Neither.

16. 4 p.c. at 102.   **17.** The 3 p.c.; £2.   **18.** $3\frac{1}{2}$ p.c. at 98.

19. $88\frac{8}{9}$.   **20.** $87\frac{1}{2}$.   **21.** $83\frac{13}{14}$.   **22.** $83\frac{4}{7}$.

23. $99\frac{7}{8}$.   **24.** $135\frac{3}{8}$.   **25.** £53 gain.   **26.** £135 gain.

27. £$206\frac{1}{4}$.   **28.** £1500.   **29.** £17. 3*s.* 9*d.*   **30.** £24,637. 10*s.*

31. £2280.   **32.** £2520.   **33.** £3377. 10*s.*; $96\frac{1}{2}$.

34. £3116.   **35.** $3\frac{1}{3}$.   **36.** £4970.

LXXXVII. Page 268.

1. $\frac{45nx}{22t}$ miles.   **2.** $\frac{88tx}{3k}$ hours.   **3.** $\frac{22mt}{45n}$ yards.

4. 3 p.m.; $24\frac{1}{2}$ m. from *A* and $17\frac{1}{2}$ m. from *B*.

5. *A*, 6 hours; *B*, $7\frac{1}{2}$ hours; 90 miles.

6. Dog, $166\frac{2}{3}$; hare, $116\frac{2}{3}$ yds.   **7.** $\frac{5}{8}$ m. and $4\frac{3}{8}$ m.

8. $11\frac{3}{7}$ hours.   **9.** $4\frac{1}{2}$ m. per hour.   **10.** 58 m.

11. $10\cdot38\frac{2}{11}$ a.m.; $24\frac{6}{11}$ m.   **12.** 198 yds.; 36 m. per hour.

13. $47\frac{8}{11}$ sec.; $5\frac{10}{33}$ sec.   **14.** 176 yds.; 45 m.

15. 2 m. per hour.   **16.** 30 m.; 50 yds.

LXXXVIII. Page 274.

1. $\frac{2}{5}$, $\frac{4}{5}$, $\frac{1}{5}$, $\frac{3}{5}$ from the starting point, then at the starting point.

2. 12 and $13\frac{1}{2}$ min.   **3.** 24 min.

4. (a) 17; (b) at points 300, 175, 50, 350, 225, 100, 400, 275, 150, 25, 325, 200, 75, 375, 250, 125 yards from the start, measured in A's direction, and at the starting point.

5. 11 meeting points; 1 hour.

6. $3\frac{3}{11}$ min. past 4, and $40\frac{4}{11}$ min. past 4.

7. $10\frac{10}{11}$ and $21\frac{9}{11}$ min. past 3.

8. 12 o'clock 60 days after; 12 hrs. 16 min.

9. $16\frac{4}{11}$ and $49\frac{1}{11}$ min. past 6; $5\frac{5}{11}$ and $38\frac{2}{11}$ min. past 10.

10. $3\frac{33}{34}$ min. past 12.

LXXXIX. Page 281.

1. 64. 2. 243. 3. 216. 4. 343. 5. 512. 6. 625.

7. 709. 8. 908. 9. 217. 10. 1039. 11. 7048. 12. 3443.

13. $2\frac{1}{4}$, $2\frac{1}{5}$, $1\frac{5}{7}$, $1\frac{7}{9}$. 14. $10\frac{13}{27}$, $15\frac{12}{29}$, $37\frac{1}{5}$.

15. 14·5, 12·03, 6·259. 16. 3·794, ·755, ·284, 1·183, ·612.

17. $\frac{a\sqrt{b}}{2b}$, $\frac{3x\sqrt{2y}}{8y}$, $a(\sqrt{2}+1)$, $2(\sqrt{3}-\sqrt{2})$.

18. $5-2\sqrt{6}$, $\frac{7+3\sqrt{5}}{2}$, $\frac{2\sqrt{a}-3\sqrt{x}}{4a-9x}$.

19. 7·872, 1·875. 20. 131 yds. 21. 3632 yds.

22. 508 yds. 23. $263\frac{3}{11}$ poles. 24. 1460 poles.

XC. Page 284.

1. ± 11. 2. ± 13. 3. $\pm\frac{7}{\sqrt{3}}$. 4. $\pm\frac{26}{\sqrt{7}}$.

5. ± 6. 6. ± 2. 7. ± 5. 8. ± 5.

XCI. Page 286.

1. 5, -2. 2. 4, 1. 3. 9, -1. 4. 5, $-\frac{17}{3}$. 5. 6, 1.

6. $1\frac{1}{2}$, -2. 7. 2, $-\frac{3}{7}$. 8. $2\frac{2}{3}$, -3. 9. 6, $-5\frac{2}{3}$. 10. 2, $-1\frac{2}{5}$.

XCII. Page 288.

1. 8 and 7. 2. 11. 3. 13 and 29. 4. 13 and 19 yds.

5. 20 miles per hour. 6. 8 sheep. 7. 50 and 24 yds.

8. 42 and 34 chains. 9. 4 yards. 10. 16 cows.

XCIII. Page 293.

1. 25. 2. 27. 3. 35. 4. 56. 5. 83.

6. 349. 7. 251. 8. 101. 9. 201. 10. 345.

11. 5642. 12. 14·3. 13. 13·1. 14. ·044. 15. ·043.

16. ·464, ·585, ·669, ·167, ·072. 17. 1·9309, 2·0928, 1·7592.

18. 3, 5, 9. 19. 4, 12. 20. 38 inches.

21. 42 inches. 22. 4489 sq. ft. 23. 12150 sq. ft.

XCIV. Page 299.

1. 77; 1025. **2.** 69; 1620.
3. $49\frac{2}{5}$, $97\frac{4}{5}$, $146\frac{1}{5}$, $194\frac{3}{5}$. **4.** 5, 9, 13, 17, 21, 25, 29.
5. ·4, ·5, ·6... **6.** -46, -44, -42... **7.** 2, 5, 8...
8. 67; 1950. **9.** $8\frac{1}{2}$. **10.** $\frac{1}{4}$.
11. $791\frac{1}{4}$ miles. **12.** 98. **13.** $12\frac{1}{2}$ miles.
14. 75*s.*; 507*s.* **15.** 7 yrs. **16.** 30 days; £14. 8*s.* 9*d.*
17. *B*, by £2000 in 20 years. **18.** £19095; 200 days.
19. 595·7 ft. **20.** 936 times.

XCV. Page 305.

1. 78732. **2.** 327680. **3.** 9. **4.** 2.
5. 16384. **6.** £204. 16*s.* **7.** ·00016.
8. 9, 27, 81, 243, 729. **9.** 3, 9, 27. **10.** $\frac{4}{25}$, $\frac{2}{5}$, 1, $\frac{5}{2}$, $\frac{25}{4}$.
11. 436905. **12.** $39\frac{9}{16}$. **13.** $\frac{1}{2}$.
14. 5461 ac.; 4096 ac. **15.** £364; £243.

XCVI. Page 311.

1. (*a*) 25 m., (*b*) 65 m., (*c*) 73 m. **2.** 7·2 inches. **3.** 85 miles.
4. 28 miles. **5.** 41 units. **6.** 25 km.
7. 15·68 chains. **8.** 117 chs.; 514·8 acres. **9.** 304 feet.
10. 7·9 hours (nearly). **11.** 48 and 20 feet. **12.** 10 and 24 cms.

XCVII. Page 314.

1. $ab+ac+ad+ae$, $a(b+c+d+e)$.
2. (*a*) 40 sq. units, (*b*) 81 sq. units, (*c*) 112 sq. units, (*d*) 19 sq. units, (*e*) 76 sq. units, (*f*) 81 sq. units.
3. (*a*) 28 sq. units, (*b*) 69 sq. units, (*c*) 80 sq. units, (*d*) 110 sq. units, (*e*) 123 sq. units, (*f*) 88 square units.
4. 75 sq. units; 15 × 5 and 25 × 3. **5.** 288 sq. ft.
6. —. **7.** —.

XCVIII. Page 317.

1. 8·4 acres (nearly). **2.** 11·06 acres (nearly).
3. 27 ch. 58 lks. **4.** 5371 sq. cms.
5. 80 and 39 metres. **6.** 55 and 48 ft.
7. 77 and 36 chains. **8.** 224 feet.

XCIX. Page 320.

1. £160. **2.** £4. 0*s.* 6*d.* **3.** 60 boards.
4. 40 boards; £65. 12*s.* 6*d.* **5.** 759 tiles. **6.** 7168 tiles.
7. 20 yds. **8.** 78 yds. 1 ft. **9.** $115\frac{1}{2}$ ft.

10. £11. 4*s*. **11.** £27. **12.** £1. 5*s*. $3\frac{3}{4}d$.
13. $85\frac{1}{3}$ yds. **14.** £28. 19*s*. **15.** 13 ft. 6 inches.
16. 2968 sq. yds. 4 sq. ft. **17.** £117. 15*s*. **18.** £176. 17*s*. 9*d*.
19. 1 shilling. **20.** £196. 10*s*. **21.** 15 ft. by 30 ft.
22. 42 ft. by 12 ft. **23.** 616 yds. and 308 yds.

C. Page 322.

1. 5 ac. 33 po. **2.** 52·26 sq. ft. **3.** $774\frac{2}{5}$ yds.
4. $9\frac{1}{2}$ chains. **5.** £64. 13*s*. 8*d*. **6.** $30\frac{1}{18}$ sq. ft.
7. 1 ac. 1 ro. 25 po. **8.** 13 ac. 1 ro. $17\frac{1}{2}$ po. **9.** $241\frac{1}{4}$ chains.
10. 6 ac. 3860 sq. yds. **11.** $178\frac{2}{3}$ poles. **12.** 2 ft. **13.** 30 yds.

CI. Page 326.

1. (*a*) 264 sq. yds., (*b*) 264 sq. yds., (*c*) 330 sq. yds., (*d*) 306 sq. metres.
2. $3\frac{1}{5}$ inches; 24 sq. inches. **3.** $28\frac{4}{5}$ yds.; 504 sq. yds.
4. 4·33 acres. **5.** 27·06 acres. **6.** 48 sq. ft.
7. 12·51 acres. **8.** 89 sq. ft. (nearly).

CII. Page 328.

1. 144 sq. yds. **2.** $437\frac{1}{2}$ sq. yds. **3.** 46·2 acres.
4. 36 yds. **5.** 58 chains. **6.** 240 yds.
7. 75 acres. **8.** 160 links. **9.** 240 links.

CIII. Page 330.

1. (*a*) 27·66 acres; (*b*) 33·83 acres; (*c*) 41·45 acres; (*d*) 39·6 acres;
(*e*) 71·75 acres; (*f*) 64·35 acres. **2.** 324 sq. yds.

CIV. Page 332.

1. 92·8 acres. **2.** 12·7 acres.
3. 19 ac. 7·675 sq. chains. **4.** 6 acres 8·73 sq. chains.

CV. Page 335.

1. 495 yds. **2.** 14 ch. 85 links. **3.** 3 ch. 85 links.
4. 11 ch. 55 links. **5.** $57\frac{3}{4}$ poles. **6.** 1386 sq. yds.
7. $3118\frac{1}{2}$ sq. yds. **8.** 4 acres 4·9064 sq. chains.
9. 2 ac. 2 ro. 12 p. 11 yds. **10.** 2464 sq. ft.
11. 6236·23 sq. yds. **12.** $115\frac{1}{2}$ po.; 363 po.
13. 12 yds. 2 ft. 6 inches; 40 yds. 1 ft. **14.** £1. 18*s*. $9\frac{1}{7}d$.
15. 88 trees. **16.** £47. 2*s*. $10\frac{2}{7}d$.
17. $9\frac{2}{7}$ miles. **18.** 4 ac. 3 ro. 35 po. $13\frac{3}{4}$ yds.

CVI. Page 337.

1. 15 sq. ft. 90 sq. inches. **2.** 8·1 sq. chains.
3. 233·64 acres. **4.** 18·62 sq. cm. **5.** 268 ac. 1 ro. 20 po.

CVII. Page 339.

1. $1642\frac{2}{3}$ sq. ft. **2.** 80 sq. yds. $1\frac{7}{8}$ sq. ft. **3.** $183\frac{1}{3}$ sq. inches.
4. 377·3 sq. ft. **5.** $30\frac{4}{5}$ ft. **6.** 72°.
7. 48·06 yds. **8.** 243 ft.

CVIII. Page 342.

1. 36. **2.** 40 chains by 32 chains. **3.** 10 chains.
4. 2·67 chains. **5.** $53\frac{1}{3}$ ch. by 40 ch. **6.** 96 gals.
7. $2\frac{1}{2}$ hours. **8.** $3\frac{1}{2}$ inches. **9.** 9801 sq. inches.
10. 1 : 660. **11.** 5. **12.** 30 ft. **13.** 56 ft.

CIX. Page 349.

1. (1) 912·82 sq. in.; (2) 1097 cub. in.; (3) 310 lbs. (app.).
2. 86 lbs. **3.** 17*s.* 4*d.* **4.** ·75 inch (nearly).
5. £7. 2*s.* 7*d.* **6.** 8 ft. **7.** $25\frac{1}{2}$ ft. long; 17 ft. broad.
8. Depth, $1\frac{1}{2}$ ft.; breadth, 3 ft.; length, 6 ft.; cost, £5. 10*s.* 3*d.*
9. 4*s.* $10\frac{29}{72}$*d.* **10.** 7 inches. **11.** 6 inches; 187 cu. ft.
12. $7\frac{6}{7}$ cu. yds. **13.** 1496 cu. ft. **14.** $1746\frac{2}{63}$ cu. yds.
15. 9*s.* 4*d.* **16.** £3. 2*s.* $10\frac{2}{7}$*d.* **17.** 3 cu. ft. 360 ins.; 19*s.* 3*d.*
18. 3 ft. $2\frac{1}{2}$ ins. **19.** 336 cu. ft. **20.** 36 cu. ft. 792 inches.
21. 1440 cu. ft. **22.** 36 cu. ft. 144 inches.
23. 3840 cu. ft. **24.** 2850 cu. yds. **25.** $18\frac{1}{3}$ cu. ft.
26. 2 ft. $9\frac{51}{77}$ ins. **27.** $48\frac{1}{8}$ cu. ft. **28.** $222\frac{1}{12}$ sq. yds.
29. $52\frac{307}{392}$ cu. ins. **30.** 411·4 gals. **31.** 48·936 sq. inches.
32. 43·874 ft. and 87·748 ft. **33.** $509\frac{1}{7}$ sq. inches.
34. 184 sq. inches. **35.** $22036\frac{6532}{7763}$ gals.

CX. Page 356.

1. 34 cu. ft. $310\frac{1}{2}$ ins. **2.** 2 cu. ft. 856 ins. **3.** 100 cu. ft.
4. $1437\frac{1}{3}$ cu. ft. **5.** $22\frac{11}{24}$ cu. ft. **6.** 744 sq. ft.
7. $27\frac{59}{63}$ yds. **8.** $326\frac{6}{7}$ yds. **9.** $3\frac{1}{7}$ cu. inches.
10. $1018\frac{2}{7}$ sq. inches. **11.** £4. 19*s.* **12.** £20. 9*s.* $2\frac{5}{7}$*d.*
13. 48 sq. yds. 6 ft. 134 ins. **14.** $381\frac{1}{3}$ cu. inches.
15. $16036\frac{4}{11}$. **16.** $381\frac{6}{7}$ cu. ins.; $254\frac{4}{7}$ sq. ins.
17. $45733\frac{1}{3}$ lbs. **18.** 3507·3 cu. ft.
19. $7\frac{19}{35}$ ozs. **20.** $75\frac{3}{7}$ sq. ft.

CXI. Page 359.

1. 25 : 16 ; 125 : 64. **2.** 9 : 49 ; 27 : 343. **3.** 13 : 8. **4.** 24 inches.
5. $42\frac{2}{3}$ lbs. **6.** 216 bushels. **7.** $4\frac{2}{3}$ inches. **8.** $1 : 3\sqrt{3}$.

Miscellaneous Examples.

I. Page 360.

1. $-\frac{1}{2}$. **2.** 4. **3.** $a+4b$. **4.** $4x^2-9y^2$.
5. $8a^3+27$. **6.** $6a^2b$; c^2x. **7.** $3a+4b-5c$.
8. $(240y-x)$ pence. **9.** $\frac{a}{b}$ miles. **10.** mn miles.
11. $(y-x)$ years. **12.** $(x-9)$ years. **13.** 2. **14.** $-\frac{1}{2}$.
15. (*a*) $x=5, y=5$; (*b*) $x=3\frac{3}{8}, y=\frac{3}{4}$. **16.** 192. **17.** 65.
18. *A*, 84 ; *B*, 42 ; *C*, 14. **19.** 147. **20.** £150.
21. (*a*) $x=9,\ y=12$; (*b*) $x=12,\ y=3$. **22.** $20x-21y-9$.
23. 28 and 12. **24.** 12 and 6. **25.** 600.
26. (*a*) $x=144,\ y=216$; (*b*) $x=2,\ y=3$. **27.** $\frac{29}{91}$.
28. $\frac{x-y}{5}$ apples. **29.** $(x-y+7)$ years.
30. $\frac{y}{x}$ pence ; $\frac{yz}{x}$ pence ; $\frac{24x}{y}$ lbs. **31.** £48. **32.** 37, 38, 39.
33. 21, 23, 25. **34.** 49 gals. **35.** Cow, £6 ; sheep, £2.
36. 20 and 12 yards. **37.** 300 acres. **38.** 45 years.
39. (1) $\frac{1}{x}, \frac{1}{y}$; (2) $\frac{x+y}{xy}$; (3) $\frac{xy}{x+y}$. **40.** $a-1,\ a,\ a+1,\ 3a$.
41. $\frac{7}{18}$. **42.** $\frac{ab}{c}$ hours. **43.** $1\frac{6}{35}$. **44.** 72 and 110.
45. 5*s*. 6*d*. and 7*s*. 4*d*. **46.** 15 men, 22 women. **47.** £4. 16*s*., £7. 4*s*.
48. 240. **49.** 562·1. **50.** $1\cdot\dot{4}2857\dot{1}$.
51. $9a^2-12ab+4b^2$, 247009. **52.** 675·1 francs.
53. £56·37. **54.** ·119140625. **55.** $13\frac{6}{7}$.
56. $(a+b-2x)$ years. **57.** ·007. **58.** ·621 mile.
59. ·994 metre. **60.** $\frac{1}{3}$ and $\frac{3}{4}$. **61.** £$\frac{abx}{1200}$.
62. $73\frac{2}{3}$ miles. **63.** $\frac{xy}{x+y}$ hours. **64.** $7\cdot56\times10^{-3}$.
65. 93·78 metres. **66.** $1\frac{35}{44}$; $\frac{2}{11}$. **67.** 4·8.
68. £895·594, £895·6. **69.** Equal. **70.** $(240a+30b+24c)$ pence.
71. $\frac{bx}{a}$ hours. **72.** $x=\frac{bc}{ad}$. **73.** Sugar, 3*d*.: tea, 1*s*. 6*d*.
74. 1*s*. $10\frac{1}{2}$*d*. **75.** 28·1638. **76.** 20 days, 30 days.
77. $\frac{2}{3}$ ton, $1\frac{1}{2}$ tons. **78.** 23·51. **79.** —. **80.** —.
81. 5·7163, 3·7163, 0·7163, $\bar{1}\cdot7163$, $\bar{3}\cdot7163$. **82.** 1,200,000 Kgs.

83. $x=\frac{3}{25}$. **84.** 42 miles. **85.** 2*s.* 8*d.*, 4*s.* 8*d.*, 11 quarts.
86. —. **87.** $\frac{100a}{b}$, $\frac{100(\pounds a - \pounds b)}{a}$. **88.** ·00123, 124.
89. 5*s.* 10*d.* **90.** 2 m., 600 yds.
91. That extension is proportional to weight ; repeat experiment.
92. $6{\cdot}57 \times 10^6$, $3{\cdot}78 \times 10^{-4}$; logs 6·8176, $\bar{4}{\cdot}5775$. **93.** 24·81.
94. $\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)$ hours. **95.** 8 and 18. **96.** —.
97. 17 and 19 cms. **98.** 8. **99.** 5·83 c.c.
100. £2·6, £3·3, £4·5, £7·5. **101.** 48 miles (approx.).
102. —. **103.** Vol. : weight = 6 : 5 ; 3·3 grams ; 3·5 c.c.
104. —. **105.** $\frac{bc}{a-b}$ hours. **106.** $\frac{pqr}{p+q}$ miles.
107. 9 and 12 miles per hour. **108.** £125 and £275.
109. 19 and 21. **110.** 27·8. **111.** 30 miles. **112.** $y=3\frac{1}{7}x$.

II. Page 368.

1. $72\frac{72}{175}$ lbs. **2.** 742 miles 746 yds. 2 ft. **3.** 599·375 grs.
4. 9·516 miles. **5.** $322666\frac{2}{3}$ cu. yds. **6.** £108 ; £112.
7. 5808 days. **8.** £32. 10*s.* ; £38. 10*s.* **9.** 4542·4801.
10. 4 min. 48·85 sec. fast. **11.** £5072. 7*s.* 10*d.* (nearly).
12. 54·468 sq. ft. and 43·91 sq. ft. **13.** 24 minutes ; 38 miles.
14. 143 miles. **15.** £96. 11*s.* 3*d.* **16.** $\frac{1}{2}$ sec.
17. 125 tons. **18.** £1147405. **19.** 28·28 ft.
20. $1367{\cdot}013\dot{8}$ yds. **21.** 14 ft. ; 3 ft. **22.** 75 lbs.
23. £675. 18*s.* 9*d.* **24.** £175. **25.** £55. 10*s.*
26. 3 dwts. $8\frac{19}{40}$ grs. **27.** 150 per cent. **28.** 48 lines.
29. 3 tons 4 cwt. 3 qrs. 4 lbs. 13 oz.
30. *A*, £424. 4*s.*; *B*, £707 ; *C*, £777. 14*s.* **31.** 10 per cent.
32. 1*s.* $0\frac{3}{4}$*d.* **33.** 144. **34.** £3600.
35. 32 lbs. **36.** 18 lbs. 10 ozs. 8 dwts. 9·273... grs.
37. $\frac{6}{19}$. **38.** 96 such chains, or 2112 yds.
39. 58 sec. **40.** ·0999. **41.** £4802. 15*s.* $3\frac{3}{4}$*d.*
42. $33\frac{1}{3}$ per cent. **43.** 14·6 in. **44.** 13 times.
45. £80. **46.** £3. 10*s.* **47.** 1*s.* 3*d.*
48. £5. **49.** 28 : 35 : 36. **50.** 28 miles.
51. 3955·7 miles. **52.** 17*s.* 6*d.* **53.** $19\frac{13}{23}$.
54. £24 loss. **55.** $1\frac{1}{15}$. **56.** £150 loss.
57. 648·9197... ac. **58.** $\frac{40}{87}$. **59.** 6400000.
60. 8·245 metres. **61.** $\frac{5}{256}$ inch. **62.** 39 ft.
63. £84. **64.** 98415 gals.

65. 204 tons 16 cwt. 1 qr. 4 lbs. **66.** £45. 12*s*. 6*d*.
67. 3744000 oranges. **68.** 1760 yds. **69.** 60 per cent.
70. £240. **71.** £19. 16*s*. 8*d*. loss. **72.** 4 days.
73. 2601 sequins. **74.** £3. 18*s*. $6\frac{2}{3}$*d*.
75. It loses $\frac{1440}{1441}$ min. in 24 hours. **76.** 880 : 623.
77. 7*s*. $7\frac{1}{2}$*d*. **78.** $93\frac{339}{532}$. **79.** 17·88 ft.
80. 4·547 litres. **81.** *A*, £9. 4*s*.; *B*, £10. 11*s*.
82. £113643. 8*s*. **83.** $\frac{200}{233}$ per cent. **84.** No ; 13*s*. 4*d*. in the £.
85. £5755. 2*s*. **86.** 16 hours. **87.** $\frac{1}{250000}$ in.
88. 7 per cent. **89.** £50. **90.** $33\frac{3}{4}$ cwts.
91. 88 p.c. **92.** $8\frac{1}{3}$ yds. **93.** 8 hours.
94. The 15 would cost £1. 16*s*. less. **95.** 29167 ; 13739 ; 3686 ; 3408.
96. Every 4 minutes. **97.** Sheep, 15*s*.; lamb, 3*s*. 4*d*.
98. £17. 3*s*. 9*d*. gain. **99.** 216 for 1 guinea. **100.** ·00319 cms.
101. 30800 cu. yds. **102.** $20\frac{8}{9}$ sq. yds. ; 4 ft. ; 1 ft. 3 in.
103. 84. **104.** 29·943... miles.
105. 1080 days 5 hrs. 55 min. $33\cdot\dot{3}$ sec.
106. At 6 p.m. May 21st same year. **107.** 45 years. **108.** £3150.
109. 2.40 p.m. ; $66\frac{2}{3}$ m. from *A* ; 1 hour 8 min. after the 2nd starts.
110. £2187. 10*s*. **111.** £2666. 13*s*. 4*d*. **112.** 3*s*.
113. 216 cubic ft. **114.** $87\frac{1}{2}$ per cent. **115.** £35. 12*s*. 3*d*.
116. Sheep, £2. 10*s*. ; pig. £1. 10*s*. **117.** 7 : 200. **118.** 13*s*. 6*d*.
119. 4·465... tons. **120.** 146388 miles 3 fur. 9 chains.
121. $5\frac{5}{7}$ miles. **122.** $75\sqrt{3}$ ft.
123. 7 tons 14 cwts. 1 qr. 14 lbs. **124.** 1*s*. 3*d*.
125. 24000 sheets. **126.** 11 : 49. **127.** £428.
128. 14 cwt. 2 qrs. $18\frac{2}{3}$ lbs. ; 3 cwts. $12\frac{4}{9}$ lbs.; 2 cwts. $24\frac{8}{9}$ lbs.
129. £248. 18*s*. 4*d*. ; £186. 13*s*. 9*d*. ; £149. 7*s*. ; £124. 9*s*. 2*d*.
130. £1240. **131.** £$71\frac{7}{23}$; 15 p.c. **132.** £7. 7*s*. 7*d*.
133. 10182·672 Kgs. **134.** 20·22 ft. **135.** £3 ; £2·64.
136. $26\frac{2}{3}$ feet. **137.** £3. 17*s*. **138.** 70 men.
139. $83\frac{1}{3}$ acres. **140.** £861. **141.** £1360. 10*s*. $10\frac{30}{49}$*d*.
142. 63·53 ; 28·27 ; 8·21. **143.** 5 per cent. ; £4000. **144.** $113\frac{1}{623}$ grs.
145. He will. **146.** 2 cwt. 22 lbs. **147.** 38 per cent.
148. $10\frac{10}{11}$ mins. past 5, and $43\frac{7}{11}$ mins. past 5. **149.** 21 min. 41 sec.
150. 6 ft. $9\frac{37}{60}$ in. or 6 ft. $9\frac{19}{30}$ in. **151.** 107 per cent.
152. 1440. **153.** $4\frac{1}{2}$ miles. **154.** $\frac{123}{427}$.
155. *A*, £1. 7*s*. 6*d*. ; *B*, £1. 13*s*. ; *C*, £2. 4*s*.
156. *A*, 6 hours ; *B*, 5 hours ; *C*, 3 hours. **157.** ·174 p.c.
158. (*a*) ·18925; (*b*) 4·866. **159.** £540. **160.** 3*s*. per lb.
161. ·9941. **162.** £10. 6*s*. 3*d*. **163.** $101\frac{169}{196}$ gals.

164. *A*, £122. 10*s.*; *B*, £157. 10*s.* **165.** £391. 7*s.* 3$\frac{3}{11}$*d.*
166. £1644. 10*s.* **167.** 2*s.* 6*d.* in the £. **168.** 26 miles per hour.
169. 1$\frac{5}{6}$ hours. **170.** 3 ft. **171.** 1·60 lbs.
172. 10959·57$\dot{4}\dot{0}$. **173.** £104. 3*s.* 3*d.*; £156. 4*s.* 10$\frac{1}{2}$*d.*
174. 11 times. **175.** 7. **176.** £90.
177. £2. 2*s.* 3*d.* **178.** 7 : 5. **179.** 18$\frac{2}{9}$ p.c.
180. 100 yards. **181.** £246. 10*s.*; 180 acres.
182. ·58 p.c. **183.** £3.; 12*s.*; £2. 8*s.*
184. *A*, £324; *B*, £648; *C*, £810; *D*, £972; *E*, £1296.
185. 11$\frac{1}{4}$*d.* **186.** 19·2 acres. **187.** 13 ft. high and 17$\frac{1}{2}$ ft. wide.
188. 302 tons 10 cwt. **189.** £10. 16*s.*
190. 8$\frac{17}{24}$ hours after starting; 28$\frac{1}{2}$ miles from *A*'s point.
191. £10376. 2*s.* 8*d.* **192.** Nov. 25th. **193.** £145.
194. 22 gals. **195.** 2 miles per hour. **196.** 54 trees; 690 ft.
197. 44·33 ac. **198.** 916·1211. **199.** 4936 yards.
200. £2. 16*s.* 0$\frac{1}{24}$*d.* **201.** 15$\frac{1}{2}$ miles (nearly). **202.** ·8$\dot{3}$.
203. 5·90625. **204.** 9 dozen at 35*s.* to 5 dozen at 21*s.*
205. 8$\frac{1}{3}$ p.c. **206.** £17. 4*s.* 0$\frac{7}{8}$*d.* **207.** £42. 10*s.*
208. 6 yards. **209.** 8680 sq. ft.
210. Boy, £3. 1*s.* 1*d.*; woman, £15. 5*s.* 5*d.*; man, £21. 7*s.* 7*d.* **211.** 5006012; 10012024.
212. 12 miles. **213.** £5. 5*s.* **214.** 80.
215. 154·3 sq. cms. **216.** 39$\frac{6}{121}$ mls. **217.** 7444$\frac{4}{5}$ lbs.
218. ·075. **219.** £431·0125.
220. 6·873 acres. **221.** 13$\frac{1}{3}$ mins.
222. 4 p.m.; 40 miles on the way; 3.30 p.m. **223.** £5. 10*s.*
224. $\frac{3}{8}$ gal. **225.** 8200·84 cub. in.; 3644·82 sq. in.
226. 4200 cub. ft. **227.** 3·48 gals. **228.** 1204·86 lbs.
229. 2·04$\dot{6}$; 22·92. **230.** 3 years. **231.** 56 ft.
232. 99. **233.** 5120 sq. ft.; 32768 cub. ft.
234. 8 lbs. of better; 32 lbs. of worse. **235.** 25·6.
236. 3$\frac{7}{11}$ miles. **237.** £240; £360; £408; £480.
238. 4 miles from one town, 6 from the other. **239.** 24.
240. £468. **241.** 75$\frac{1}{6}$ yards. **242.** 28$\frac{4}{7}$ days.
243. 616 yds.; 308 yds. **244.** 120000 men. **245.** 82$\frac{1}{7}$.
246. 17$\frac{1}{9}$ days. **247.** 750 men. **248.** 25$\frac{3}{4}$ yds; 30$\frac{9}{10}$ yds.
249. £12. **250.** £900; £1120; £750.
251. 49·851 chains. **252.** Income, £900; rate, 5*d.*
253. 117$\frac{1}{2}$ yds., £39. 13*s.* 1$\frac{1}{2}$*d.* **254.** £4. 18*s.*
255. 173·2 cub. in. **256.** 9 ft. **257.** 25 in.
258. 36$\frac{2}{3}$ ft. **259.** 9 days. **260.** 6$\frac{2}{3}$ per cent.

261. *B*, $\frac{3}{8}$; *C*, $\frac{5}{8}$. **262.** £168. **263.** 5745144.
264. 144 lbs. avoir; 175 lbs. troy. **265.** $12\frac{4}{5}$ ft.
266. 19,800,000 gals. **267.** 45 mins. $10\frac{10}{17}$ secs. past 10.
268. 29·068 acres. **269.** 66350·59 cub. in. **270.** 4 ft.
271. Slow, 20 miles; fast, 32 miles. **272.** £74.
273. 1 ton 15 cwt. 51·625 lbs. **274.** 14·34 cub. ft.
275. $7\frac{1}{5}$ hours. **276.** 4 ac. 1 ro. 5·792 po.
277. 10 oranges. **278.** 20 days. **279.** $76\frac{1}{2}$ inches.
280. 120 men. **281.** £646. 10*s*. $4\frac{1}{2}$*d*. **282.** ·0469 cm.
283. 1035·638... grms. **284.** $\frac{7}{16}$. **285.** ·76536....
286. 900 members; 4*d*. extra. **287.** $\frac{7}{30}$.
288. $150\frac{1}{2}$ yds. **289.** 14 seconds. **290.** —.
291. £1070. **292.** £39. 11*s*. 8*d*. **293.** £150.
294. $3\frac{11}{18}$ hours. **295.** £96. 6*s*. **296.** $107\frac{13}{18}$ sq. ft.
297. $119\frac{739}{1144}$. **298.** $25\frac{1}{8}$ p.c. **299.** 7·21721 acres.
300. £4. 17*s*. 6*d*. **301.** £500. **302.** 892·92 yds.
303. 14*s*. 7*d*. **304.** 36787666500.
305. £3430; £4573. 6*s*. 8*d*.; £5716. 13*s*. 4*d*. **306.** 225 horses.
307. $10\frac{1}{2}$ days; *A*, $5\frac{5}{8}$ ac.; *B*, $6\frac{1}{2}$ ac. **308.** 34 sq. ft. $132\frac{4}{7}$ ins.
309. £112. 12*s*. $1\frac{23}{25}$*d*. **310.** 5*s*. 10*d*. **311.** £455. 8*s*.
312. 4*s*. 2*d*. **313.** $\frac{3}{4}$*d*.
314. 16 m. 41 ch. 20 lks.; 19 m. 65 ch. 44 lks.; 23 m. 9 ch. 68 lks.
315. $\frac{16}{85}$ mile per hour. **316.** 6 chs.; $2\frac{1}{2}$ chs. **317.** 282150 gals.
318. 9 tons. **319.** $633\frac{3}{5}$ lbs. **320.** 25 per cent.
321. $91\frac{1}{2}$. **322.** £100; 5 p.c. **323.** £17. 3*s*. 4*d*.
324. 4*s*. 8*d*. **325.** 156 boys. **326.** 10296 sq. ft.; 125 ft.; 82·37 ft.
327. 209·58 sq. yds. **328.** $1\frac{3}{5}$ hrs.; $22\frac{2}{5}$ and $17\frac{3}{5}$ miles; $1\frac{1}{5}$ hrs.
329. £2 10*s*. **330.** 2·309.... **331.** £50820.
332. 943·34625 cu. ft. **333.** 1800; 2000 francs.
334. *A*, 13 francs; *B*, 18 francs; *C*, 20·7 francs. **335.** 35·496 ft.
336. £122. 10*s*. **337.** 38·11 and 76·22 yds.; 30·4 yds.
338. 133·785 ft. **339.** $£x \times \frac{2p+2q-2a+a\sqrt{2}}{66}$.
340. 6 ft. $6\frac{3}{4}$ inches; $\frac{45}{148}$ inch per hour. **341.** 825 lbs.
342. $\cdot\dot{4}\dot{5}$. **343.** 9 ft. **344.** 468 ft.
345. 168 cub. in.; 760 bullets. **346.** 13*s*. 11·635... *d*.
347. 52 Chinamen. **348.** 7·745. **349.** 220 sq. ft.
350. £217. 5*s*. $8\frac{4}{7}$*d*. **351.** 22·48 ins.; 1588 sq. ins.
352. $1\frac{1}{2}$ tons and $2\frac{1}{2}$ tons. **353.** $\frac{px}{100}+\frac{qy}{100}=\frac{z}{100}$ of $(p+q)$.

354. £25483. 10*s*. **355.** B.P.—horse, £40 ; carriage, £60 ;
S.P.—horse, £60 ; carriage, £45 ;
356. 25 gals. **357.** £$46\frac{2}{3}$. **358.** £1200 and £1300.
359. 1848 yds. **360.** 35 yrs. ; 25 yrs. **361.** £1500.
362. 8250 ft. **363.** 322 yds. 2 ft. **364.** 4 lbs.
365. $\frac{45}{49}$ hour. **366.** 3 fr. 84 centimes. **367.** £2400.
368. $9\frac{102}{179}$ minutes. **369.** 20·5 secs. **370.** 14·089896 inches.
371. £67. 10*s*. ; $37\frac{1}{2}$ p.c. **372.** 42,768,000 gals. **373.** £2. 8*s*.
374. £10. 4*s*. loss. **375.** 67·87... ft. **376.** 138·56 acres.
377. 37·04 lbs. **378.** 35202·44 inches. **379.** $10392\cdot0246704\dot{4}\dot{5}$.
380. $16\frac{4}{5}$ gals. **381.** 157380 : 523. **382.** $1\frac{4}{13}$ hours.
383. 33 lbs. **384.** 5 miles. **385.** $10\frac{1}{2}$ p.c.
386. 6·684... inches. **387.** 42·426... yds. **388.** 225.
389. 5·201. **390.** $1\frac{1}{4}$ cu. ft. **391.** £500 each.
392. Height, 15 ft.; br., 30 ft.; length, 60 ft.; cost, £82. 10*s*.
393. Oxygen, 1·105... ; nitrogen, ·972.... **394.** £126 profit.
395. Antwerp ; 4*s*. $7\frac{201}{4210}$*d*. per qr. **396.** 76·2001....
397. 3*s*. 8*d*. **398.** 999434·37... dynes. **399.** £147. **400.** £125 ; £250.
401. $2\frac{1}{2}$ inches. **402.** *A*, 60 ; *B*, 120 ; *C*, 80 days.
403. £5. 16*s*. 8*d*. **404.** £1260 ; *A*, £420 ; *B*, £540 ; *C*, £300.
405. £6500. **406.** $\frac{5}{9}$. **407.** *A*, £230 ; *B*, £300. **408.** 1512 yds.
409. $A : B = 40 : 53$. **410.** 5 ft. $9\frac{1}{2}$ inches ; 23 ft. 2 inches.
411. 2·315 hrs. (nearly). **412.** 864. **413.** 7 : 1.
414. 20 ozs.; 3 : 1. **415.** £4. 0*s*. 3*d*.; 7 per cent.
416. 2 miles an hour. **417.** 40° below zero. **418.** $6\frac{13}{14}$ days.
419. 20 yds.; 3*s*. 6*d*. per yd. **420.** *A*, 855 ; *B*, 684 ; 57 in a row.
421. 28 : 15. **422.** $\frac{1}{7}$. **423.** £60 ; £50. **424.** $\frac{5}{26}$.
425. 52 miles ; 28 miles. **426.** $4\frac{1}{2}$ ft. **427.** £1 = 25·92 francs.
428. *A*, 30 years ; *B*, $31\frac{1}{20}$ years. **429.** $6\frac{2}{3}$° ; 241·25°.
430. 25 ft. 11 inches. **431.** £837. 5*s*. 2·413... *d*. **432.** 352 of each kind
433. 44989·77... francs. **434.** 122 florins. **435.** $6\frac{3}{17}$ per cent.
436. £58. 16*s*. **437.** Express, 7.40 ; slow train, 8.15 p.m.
438. 6 days from commencement. **439.** £2. 10*s*.
440. £185. **441.** 48. **442.** $\frac{3}{4}$ of it ; $\frac{1}{4}$ each time.
443. 60 miles ; 7.30 a.m.; $2\cdot52\frac{1}{2}$ p.m.; 8.30 p.m.; 3.30 p.m.
444. 2·8 inches ; 9·4582 sq. inches. **445.** £1564.
446. £4875 ; £1625 ; £1083. 6*s*. 8*d*.; £866. 13*s*. 4*d*.; £780.
447. $126\frac{63}{275}$ grs. **448.** *A*, $7\frac{1}{2}$ miles ; *B*, $4\frac{1}{2}$ miles.
449. 5 miles, and $3\frac{1}{2}$ miles per hour. **450.** *B*, 6 sec.; *C*, 15 sec.
451. 35 yrs.; 5 yrs. **452.** 4880 yds. **453.** 85 wine ; 35 water.
454. 28 days. **455.** *A*, £94. 10*s*.; *B*, £168 ; *C*, £396.

456. £500. **457.** 330 yds.; 1650 yds.
458. *A*, £12375; *B*, £17750. **459.** 9 days.
460. $32\frac{1}{2}$ ft. **461.** 15 ft.; 20 ft. **462.** $2\frac{8}{11}$ sec.
463. 25 miles; 100 miles. **464.** 12 : 13. **465.** £3600.
466. £32175. **467.** 13; £3. 5*s*. **468.** 405 yds.; 805 yds.
469. 45 min.; 90 min. **470.** 17; £1. 0*s*. 4*d*. **471.** 22 gals.; 10 gals.
472. 246073 : 237600. **473.** —. **474.** 265. **475.** $20\frac{19}{73}$.
476. ·24. **477.** *A*, £18400; *B*, 12266. 13*s*. 4*d*. **478.** 124 fathoms.
479. £1. 15*s*.; £4. 7*s*. 6*d*. **480.** 9 sq. yds.; 25 sq. yds.
481. £26 8*s*. **482.** 6 inches.
483. *A*, 147 sec.; *B*, 168 sec. **484.** 36 inches; 72 inches.
485. 35·38 ozs. (nearly). **486.** $\frac{20}{21}$ second.
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